CYCLIC FORMATION CONTROL FOR SATELLITE FORMATION USING LOCAL RELATIVE MEASUREMENTS

Baolin Wu,∗ Danwei Wang,∗ and Eng Kee Poh**

Abstract

This paper introduces a cyclic formation control design for satellite formation using circulant matrix. Each satellite controls itself by using only the local relative states from itself to its two neighbouring satellites. The local relative states can be obtained by local line-of-sight measurements. The formation control problem with n satellites is thus reduced into stabilization problem of a single satellite with the same relative dynamics, individualized by a specific scalar. The scalar takes the values of nonzero eigenvalues of a circulant matrix representing the topology of relative measurements among satellites. The modified relative dynamics can be viewed as linear system with uncertain parameters and an optimal guaranteed cost control law can be applied. Simulation results are presented to show the efficiency of the proposed cyclic formation control for satellite formation flying.

Key Words

Cyclic formation control, circulant matrix, satellite formation

1. Introduction

Satellite formation flying needs precise co-ordination among multiple satellites whose dynamics are coupled through a common control law [1]. It has attracted much attention recently due to its perceived operational benefits, such as reduced cost, mission flexibility, improved performance, increased reliability and enhanced survivability. However, there are several technical challenges that formation flying missions face: high-precision relative navigation, distributed communication, trajectory optimization [2] and formation control [1].

Most control strategies for satellite formation use leader–follower architecture in which one satellite is designated as leader and the remaining satellites are maneuvered to follow the same leader. Numerous such contributions differ primarily in the types of formation tracking control laws designed for the followers. Most modern control paradigms have been investigated to solve this problem, such as linear quadratic regulator [3]–[6], neural adaptive control law [7], differential-geometric methods [8], model predictive control law [9], linear matrix inequalities-based control [10], sliding model techniques-based nonlinear tracking control [11] and Lyapunov-based nonlinear control law [12], [13]. Several decentralized control approaches are also proposed in [14]–[17]. A survey of various control design methodologies for satellite formation can be found in [1]. Most of these studies suggested the use of global positioning system (GPS) as measurement hardware required for relative navigation in formation flying missions [14], [18].

However, GPS is not suitable for precision relative navigation in formation flying missions beyond low Earth orbits. Also, full collaboration and communication architecture among the formation satellites is not preferable for some formation flying missions due to high operational cost and technology immaturity. These two reasons motivate us to find an alternative to GPS-based relative navigation for formation flying missions. Gurfil [18] developed a cyclic formation control law based on line-of-sight (LOS) measurements only. However, bounded formation, which is obtained in the condition that the energies of satellites in the formation are matched, is not sufficient to satisfy all the requirements for satellite formation flying missions. However, in some missions, the relative positions and velocities between each of the satellites need to be controlled precisely.

This paper develops a cyclic formation control approach for satellite formation using only local relative measurements. In the proposed approach, each satellite controls itself by using only the local relative states from itself to its two neighbouring satellites. The local relative states can be obtained by local LOS measurements. By using circulant matrix and Kronecker algebra, the formation control problem with n satellites is reduced into stabilization problem of a single satellite with the same relative dynamics, individualized by a specific scalar. The scalar takes the values of nonzero eigenvalues of a circulant matrix representing the topology of relative measurements among the
satellites. The modified relative dynamics can be viewed as linear system with uncertain parameter. Optimal guaranteed cost control law is then chosen to stabilize the resulting linear system with uncertain parameters, because the optimal guaranteed cost control law can stabilize the linear norm-bounded uncertain system while minimizing the associated LQR cost function [19]. The choice of the weight matrices in the associated LQR cost function is a tradeoff between fuel consumption and convergence time.

The proposed cyclic formation control architecture has several advantages over leader–follower formation control architecture. Firstly, it needs only local relative measurements. GPS-based relative position sensors or GPS-like pseudolite system are not required. Furthermore, no infrastructure for full collaboration and communication among the formation satellites is required. Secondly, the proposed decentralized control architecture is completely decentralized in the sense that there is neither a coordinating agent nor instability resulting from single point failures. At last, the proposed decentralized control architecture is scalable. The controller does not need to be redesigned or even modified when satellite joins in or leaves the original formation flying system.

This paper is organized as follows. In Section 2, CW equations describing the relative motion is introduced and mathematical preliminaries of circulant matrices and Kronecker algebra are reviewed. Section 3 presents problem formulation of cyclic formation control. Section 4 derives the condition under which the goal of formation control is achieved. Meanwhile, the relative state of each satellite after stabilization is derived. In Section 5, optimal guaranteed cost controller is designed for the resulting linear system with parameter uncertainty. Section 6 presents comprehensive simulation results to validate the proposed formation control method using MATLAB and satellite tool kit (STK).

2. Formation Dynamics and Mathematical Preliminaries

This section reviews CW equations which describe relative motion of satellite with respect to a circular reference orbit. Then some results from the theory of circulant matrices and Kronecker algebra are provided as the preliminaries to the proposed formation control in the following sections. For details, readers are referred to [18], [20].

2.1 Relative Motion Dynamics

The relative motion of a satellite with respect to a circular reference orbit can be described by the well-known CW equations [21]. Figure 1 shows a local vertical local horizontal (LVLH) frame with its origin on the reference orbit and with a co-ordinate system spanned by the following unit vectors:

\[ \mathbf{x} = \frac{\mathbf{r}}{r}, \quad \mathbf{z} = \frac{\mathbf{h}}{h}, \quad \mathbf{y} = \mathbf{z} \times \mathbf{x} \]  \hspace{1cm} (1)

where \( \mathbf{r} \) and \( r = |\mathbf{r}| \) are the position vector and the geocentric distance of reference orbit, respectively, \( \mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} \) and \( h = |\mathbf{h}| \) are the reference orbit’s angular momentum vector and its magnitude, respectively.

The CW equations are given as:

\[ \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \]  \hspace{1cm} (2)

\[ A = \begin{bmatrix}
0 & 3\omega_c^2 & 2\omega_c & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
-2\omega_c & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -\omega_c^2 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad B = \begin{bmatrix} 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ \end{bmatrix}^T \]  \hspace{1cm} (3)

where \( \mathbf{x}(t) = [\dot{x}, x, \dot{y}, y, \dot{z}, z] \) represent relative positions and velocities of the satellite with respect to the LVLH frame, \( \omega_c \) is the natural frequency of the reference orbit, \( e \) is the eccentricity, \( \mathbf{u}(\theta) = [u_x, u_y, u_z]^T \) represent the control forces acting on the satellite.

2.2 Circulant Matrices [18]

A circulant matrix of order \( n \) is a square matrix of the form:

\[ C = \begin{bmatrix}
c_1 & c_2 & \cdots & c_{n-1} \\
c_n & c_1 & \cdots & c_{n-2} \\
\vdots & \vdots & \ddots & \vdots \\
c_2 & c_3 & \cdots & c_1
\end{bmatrix} =: circ[c_1, c_2, \ldots, c_n] \]  \hspace{1cm} (4)

Each subsequent row is simply the row above shifted one element to the right and wrapped around, \( i.e., \) modulo \( n \). The matrix is determined by the first row.
An important property of circulant matrices is their diagonalizability which will be shown here. Define $\xi = e^{2\pi j/n}$, with $j = \sqrt{-1}$, and let $F_n$ denote the $n \times n$ Fourier matrix given below:

$$F_n = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 & \cdots & \xi^{n-1} \\ 1 & \xi^2 & \xi^4 & \cdots & \xi^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi^{n-1} & \xi^{2(n-1)} & \cdots & \xi^{(n-1)(n-1)} \end{bmatrix}$$

(5)

$F_n^*$ denotes the conjugate transpose of $F_n$, and it is clear $F_n^* F_n = I_n$. An $n \times n$ circulant matrix $C$ is diagonalizable with the Fourier matrix $F_n$ by $\Lambda_n = F_n^* C F_n$, where

$$\Lambda_n = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

(6)

and the eigenvalues of $C$ are given by,

$$\lambda_i = p(\xi^{i-1}), \ \forall \ i = 1, 2, \ldots, n$$

(7)

and $p$ is a polynomial called the circulant’s representer given by,

$$p(\xi) = c_1 + c_2 \xi + c_3 \xi^2 + \cdots + c_n \xi^{n-1}$$

(8)

### 2.3 Kronecker Algebra

Kronecker algebra is a useful tool for modelling and manipulating equations which govern formation motion [20]. If $A$ is an $m \times n$ matrix and $B$ is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is the $mp \times nq$ block matrix as follows:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

(9)

If $\dot{x}_i(t) = Ax_i(t) + Bu_i(t)$, $i = 1, 2, \ldots, n$, represents the relative motion dynamics of a satellite with respect to the reference orbit, then the dynamics of $n$ satellites can be represented by $\dot{x}(t) = (I_n \otimes A)x(t) + (I_n \otimes B)u(t)$. Another important case is if $L_{n \times n}$ is a matrix representing the topology of scalar data exchange from $n$ satellites, then this information exchange of a vector data of dimension $m$ from $n$ satellites can be represented by concatenating the $n$ vectors of dimension $m$ into a single vector of dimension $nm$ and multiplying it by $I_{n \times n} \otimes I_m$.

The following equations are some important properties of Kronecker product:

$$ (A \otimes B)(C \otimes D) = AC \otimes BD $$

(10)

$$ (A \otimes B)^T = A^T \otimes B^T $$

(11)

$$ (A \otimes B)^{-1} = A^{-1} \otimes B^{-1} $$

(12)

### 3. Formulation of Cyclic Formation Control

Consider a formation of $n$ satellites with identical dynamics. The relative motion dynamics of each satellite with respect to circular reference orbit is modeled in (2). Now it can be rewritten as follows with index:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \ \forall \ i = 1, 2, \ldots, n \quad (13)$$

Motivated by the study of cooperative control for multi-agent systems using local interaction [20], [22], this section proposes a cyclic formation control approach in which each satellite controls itself by using only the local relative states from itself to its two neighbouring satellites. The local relative states can be obtained by local LOS measurements.

The proposed cyclic formation control is to design the following decentralized state-feedback control for each satellite:

$$u_i(t) = K[\Delta x_{i-1}(t) + \Delta x_{i+1}(t) - \Delta x_{i,i-1}^d(t) - \Delta x_{i,i+1}^d(t)], \ \forall \ i = 1, 2, \ldots, n \quad (14)$$

where the subscripts $n + 1$, 1, denote the same satellite. $K$ is a constant state-feedback gain matrix. $\Delta x_{i,i-1} = x_{i-1} - x_i$ and $\Delta x_{i,i+1} = x_{i+1} - x_i$, denote, respectively, the relative states of the $(i+1)$th satellite and the $(i-1)$th satellite with respect to the $i$th satellite. $\Delta x_{i,i+1}^d = x_{i+1}^d - x_i^d$ and $\Delta x_{i,i-1}^d = x_{i-1}^d - x_i^d$, respectively, denote the desired relative states of the $(i+1)$th satellite and the $(i-1)$th satellite with respect to the $i$th satellite. $x_i$ and $x_i^d$ denote, respectively, the actual and the desired relative states of the $i$th satellite with respect to reference orbit.

**Remark 1.** The local relative state between satellites $\Delta x_{i,j} \forall \ i,j = 1,2,\ldots, n$ can be obtained by local LOS measurement. The topology of relative measurements among satellites corresponding to the control law in (14) is cyclic, which is illustrated in Fig. 1. This cyclic topology gives rise to the concept of cyclic formation control. The goal of formation control is

$$\lim_{t \to \infty} (\Delta x_{i,j}(t) - \Delta x_{i,j}^d(t)) = 0, \ \forall \ i,j = 1, 2, \ldots, n \quad (15)$$

As observed from the above equation, the desired relative positions and velocities between each of the satellites are obtained. Note that the relative states $\Delta x_{i,j}$ between satellites in formation need to be controlled, but not necessarily the relative states $x_i$ of satellites with respect to the reference orbit.

**Assumption 1.** The desired relative states between satellites $\Delta x_{i,j}^d$ are assumed to be also governed by the relative motion dynamics in (2) without control inputs as follows:

$$\Delta x_{i,j}^d(t) = A\Delta x_{i,j}^d(t), \ \forall \ i,j = 1, 2, \ldots, n \quad (16)$$

For satellite formation control, this assumption is easily satisfied. For instance, the above assumption holds for
Figure 2. Illustration of line-of-sight measurements among satellites.

both of circular formation and projected circular formation designed in [21].

**Remark 2.** LOS measurements illustrated in Fig. 2 include relative distance measurement and relative orientation measurements in (17). LOS measurements can be acquired by several readily available methods, such as the use of vision-based sensors that combine star trackers and optical/RF ranging [18], [23]. Note that no absolute measurement such as inertial position and velocity of each satellite is required in the proposed control theme in (14), and only local relative measurement is needed.

\[
\begin{align*}
\Delta r_{i,j} &= \sqrt{\Delta x_{i,j}^2 + \Delta y_{i,j}^2 + \Delta z_{i,j}^2} \\
\Delta \theta_{i,j} &= \arcsin(\Delta z_{i,j}/\Delta r_{i,j}) \\
\Delta \phi_{i,j} &= \arctan(\Delta y_{i,j}/\Delta x_{i,j})
\end{align*}
\]

(17)

\[\forall i, j = 1, 2, \ldots, n\]

4. Cyclic Formation Control Design

This section derives the condition under which the goal of formation control is achieved. Meanwhile, the relative state of each satellite in steady state is given.

The cyclic formation control law given in (14) can be re-written as:

\[
u_i(t) = K[(x_{i-1}(t) + x_{i+1}(t) - 2x_i(t)) - (x_i^d(t) + x_{i+1}^d(t) - 2x_i^d(t))], \quad \forall i = 1, 2, \ldots, n
\]

(18)

where subscripts \(n + 1\), 1 denote the same satellite, subscripts \(n\), 0 denote the same satellite. \(x_i\) and \(x_i^d\) denote, respectively, the actual and the desired relative states of the \(i\)th satellite with respect to reference orbit.

Collecting the equations for all satellites into a single system using Kronecker product, the system in (13) and (18) are represented as follows:

\[
\begin{align*}
\dot{x}(t) &= (I_n \otimes A)x(t) + (I_n \otimes B)u(t) \\
u(t) &= (I_n \otimes K)(L_G \otimes I_6)(x^d(t) - x(t))
\end{align*}
\]

(19) (20)

where, \(x = [x_1, x_2, \ldots, x_n]^T\), \(x^d = [x_1^d, x_2^d, \ldots, x_n^d]^T\), \(u = [u_1, u_2, \ldots, u_n]^T\), \(L_G\) is a circulant matrix presenting the topology of relative measurements among satellites in a formation.

**Theorem 1.** Consider a satellite formation described by (13) with the control law in (14). Suppose Assumption 1 is valid and if the control gain \(K\) in (14) can be chosen such that the following system is asymptotically stable.

\[
\chi(t) = (A - \lambda_i B K)\chi(t) \quad \forall i = 2, 3, \ldots, n
\]

(21)

where \(\chi\) are the state variable of appropriate dimensions. \(\lambda_i\) are nonzero eigenvalues of \(L_G\), and \(\lambda_i = 2 - \cos(2\pi(i - 1)/n), \forall i = 2, 3, \ldots, n\), the goal of formation control defined in (15) is achieved.

Furthermore, the relative state of each satellite after the goal of formation control is achieved is given by, with \(x_i^f = x_i^d - x_i\),

\[
x_{i,ss}(t) = \lim_{t \to \infty} \left( x_i^f(t) - \frac{1}{n} \sum_{i=1}^{n} (e^{At}x_i^f(0)) \right), \quad \forall i = 1, 2, \ldots, n
\]

(22)

**Proof:** Note that the systems described by (13) with the control law in (14) can be represented in a single system in (19) with control law in (20) using the Kronecker product. Substituting (20) into (19) yields:

\[
x(t) = (I_n \otimes A)x(t) + (I_n \otimes BK)(L_G \otimes I_6)(x^d(t) - x(t))
\]

(23)

Assumption 1 implies:

\[
x^f(t) = x^d(t) - x(t)
\]

(24)

Define:

\[
x^f(t) = x^d(t) - x(t)
\]

(25)

Subtracting (24) from (23), and using the definition in the above equation lead to,

\[
\dot{x}^f(t) = (I_n \otimes A)x^f(t) - (I_n \otimes BK)(L_G \otimes I_6)x^f(t)
\]

(26)
According to (4), the matrix $L_G$ is circulant with the generating row $[2, -1, 0, \ldots, 0, -1]$. Citing the diagonalizability property of a circulant matrix presented in the first section, the matrix $L_G$ can be diagonalized as follows:

$$
\Lambda_L = F_n^* L_G F_n = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)
$$

where $\lambda_i, \forall i = 1, 2, \ldots, n$ are eigenvalues of $L_G$, which are given by,

$$
\lambda_i = 2 - 2 \cos(2\pi(i - 1)/n), \quad \forall i = 1, 2, \ldots, n
$$

(28)

Because $\Lambda_L = F_n^* L_G F_n$, $F_n \otimes I_6$ can transform $L_G \otimes I_6$ into $\Lambda_L \otimes I_6$ as follows using the properties of the Kronecker product in (10) and (11).

$$(F_n \otimes I_6)^*(L_G \otimes I_6)(F_n \otimes I_6) = \Lambda_L \otimes I_6
$$

(29)

Let $\hat{\mathbf{x}} = (F \otimes I_6) \mathbf{x}$, by using the properties of Kronecker product in (10)–(12), (26) can be rewritten as:

$$
\frac{d}{dt} \hat{\mathbf{x}}^e(t) = (I_n \otimes A) \hat{\mathbf{x}}^e(t) - (I_n \otimes BK)(\Lambda_L \otimes I_6) \hat{\mathbf{x}}^e(t)
$$

(30)

The right-hand side of the above equation is block-diagonal, and its diagonal blocks are of the form:

$$
\dot{x}^e_i(t) = A \ddot{x}^e_i(t) - \lambda_i BK \ddot{x}^e_i(t), \quad \forall i = 1, 2, \ldots, n
$$

(31)

So the state-transition matrix of the transformed system in (30) can be written as:

$$
\phi_{\hat{\mathbf{x}}^e}(t) = \text{diag}(e^{At}, e^{(A - \lambda_2 BK)t}, \ldots, e^{(A - \lambda_n BK)t})
$$

(32)

If the closed-loop system in (21) is asymptotically stable,

$$
\lim_{t \to \infty} e^{[A - \lambda_i BK]t} = 0_{6 \times 6}, \quad \forall i = 2, 3, \ldots, n
$$

(33)

Transforming state variable $\hat{\mathbf{x}}^e$ back into the original system and noting that the first row (column) of the Hermitian matrix $F_n$ is $[1, 1, \ldots, 1]_n$ in (5) leads to

$$
\mathbf{x}^e_{ss}(t) = (F_n \otimes I_6) \lim_{t \to \infty} \phi_{\hat{\mathbf{x}}^e}(t)(F_n \otimes I_6)^* \mathbf{x}^e(0)
$$

$$
= (F_n \otimes I_6) \lim_{t \to \infty} \text{diag}(e^{At}, 0, \ldots, 0)(F_n \otimes I_6)^* \mathbf{x}^e(0)
$$

$$
= (F_n \otimes I_6) \lim_{t \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} (e^{At} \mathbf{x}^e_i(0)) \right)
$$

(34)

That is

$$
\mathbf{x}^e_{ss}(t) = \lim_{t \to \infty} \frac{1}{n} \sum_{i=1}^{n} (e^{At} \mathbf{x}^e_i(0)), \quad \forall i = 1, 2, \ldots, n
$$

(35)

So the relative state of each satellite at steady states after the goal of formation control is achieved is

$$
\mathbf{x}_{i,ss}(t) = \lim_{t \to \infty} (\mathbf{x}^e_i(t) - \mathbf{x}^e_{i,ss}(t))
$$

$$
= \lim_{t \to \infty} \left( \mathbf{x}^e_i(t) - \frac{1}{n} \sum_{i=1}^{n} (e^{At} \mathbf{x}^e_i(0)) \right)
$$

(36)

Using the above equation leads to

$$
\lim_{t \to \infty} \Delta \mathbf{x}_{i,j}(t) = \mathbf{x}_{j,ss}(t) - \mathbf{x}_{i,ss}(t)
$$

$$
= \left( \lim_{t \to \infty} \mathbf{x}^e_j(t) - \mathbf{x}^e_{j,ss}(t) \right) - \left( \lim_{t \to \infty} \mathbf{x}^e_i(t) - \mathbf{x}^e_{i,ss}(t) \right)
$$

$$
= \left( \lim_{t \to \infty} \mathbf{x}^e_j(t) - \mathbf{x}^e_i(t) \right)
$$

$$
= \lim_{t \to \infty} \Delta \mathbf{x}^e_{i,j}(t), \quad \forall i, j = 1, 2, \ldots, n.
$$

(37)

where $\Delta \mathbf{x}^e_{i,j}(t)$ is the desired relative state of the $i$th satellite with respect to the $j$th satellite. As concluded from (37), the goal of the formation defined in (15) is achieved.

Now, the standing problem is now to design state feedback gain $K$ such that the system in (21) is asymptotically stable. The closed-loop system in (21) can be viewed as the following linear parameter uncertain system with state feedback control.

$$
\begin{cases}
\dot{\chi}(t) = A \chi(t) - \lambda_i B \upsilon(t); \\
\upsilon(t) = K \chi(t), \quad \forall i = 2, 3, \ldots, n
\end{cases}
$$

(38)

The optimal guaranteed cost control law to be introduced in Section 5 is chosen to design the above state feedback controller, because optimal guaranteed cost control law can stabilize the linear system with uncertain parameter while minimizing the LQR cost function. The choice of the weight matrices in the cost function is a tradeoff between the fuel consumption and the convergence rate.

**Remark 3.** The zero eigenvalue of $L_G$ can be interpreted as the uncontrollable mode of absolute motion of the satellite formation. Since $\mathbf{x}_d$ is a known function and does not affect controllability, then dropping $\mathbf{x}_d$ from (23) leads to the following:

$$
\mathbf{x}(t) = \begin{bmatrix}
A - 2BK & BK & 0 & 0 & \cdots & 0 & BK \\
BK & A - 2BK & BK & 0 & \cdots & 0 & 0 \\
0 & BK & A - 2BK & BK & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
BK & 0 & 0 & 0 & \cdots & BK & A - 2BK
\end{bmatrix}_{6n \times 6n}
$$

(39)

Use the change of coordinates $\mathbf{x} = P \mathbf{x}$ such that

$$
\dot{\mathbf{x}}_1 = \mathbf{x}_1, \dot{\mathbf{x}}_2 = \mathbf{x}_2, \quad \dot{\mathbf{x}}_{n-1} = \mathbf{x}_{n-1}, \quad \dot{\mathbf{x}}_n = \sum_{i=1}^{n} \mathbf{x}_i
$$

(40)

which gives, with some manipulations,
According to (4), the matrix $L_G$ is circulant with the generating row $[2, -1, 0, \ldots, 0, -1]$. Citing the diagonalizability property of a circulant matrix presented in the first section, the matrix $L_G$ can be diagonalized as follows:

$$
\Lambda_L = F_n^* L_G F_n = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)
$$

(27)

where $\lambda_i, \forall i = 1, 2, \ldots, n$ are eigenvalues of $L_G$, which are given by,

$$
\lambda_i = 2 - 2 \cos(2\pi(i - 1)/n), \quad \forall i = 1, 2, \ldots, n
$$

(28)

Because $\Lambda_L = F_n^* L_G F_n$, $F_n \otimes I_6$ can transform $L_G \otimes I_6$ into $\Lambda_L \otimes I_6$ as follows using the properties of the Kronecker product in (10) and (11).

$$(F_n \otimes I_6)^*(L_G \otimes I_6)(F_n \otimes I_6) = \Lambda_L \otimes I_6
$$

(29)

Let $\tilde{x} = (F \otimes I_6)x$, by using the properties of Kronecker product in (10)–(12), (26) can be rewritten as:

$$
\dot{\tilde{x}}^e(t) = (I_n \otimes A)x^e(t) - (I_n \otimes BK)(\Lambda_L \otimes I_6)\tilde{x}^e(t)
$$

(30)

The right-hand side of the above equation is block-diagonal, and its diagonal blocks are of the form:

$$
\dot{x}_i^e(t) = A\dot{x}_i^e(t) - \lambda_i BK x_i^e(t), \quad \forall i = 1, 2, \ldots, n
$$

(31)

So the state-transition matrix of the transformed system in (30) can be written as:

$$
\phi_{\tilde{x}}^e(t) = \text{diag}(e^{At}, e^{-(A-\lambda_2 BK)t}, \ldots, e^{(A-\lambda_n BK)t})
$$

(32)

If the closed-loop system in (21) is asymptotically stable,

$$
\lim_{t \to \infty} e^{(A-\lambda_i BK)t} = 0_{6 \times 6}, \quad \forall i = 2, 3, \ldots, n
$$

(33)

Transforming state variable $\tilde{x}^e$ back into the original system and noting that the first row (column) of the Hermitian matrix $F_n$ is $[1, 1, \ldots, 1]_n$ in (5) leads to

$$
x_{i,ss}^e(t) = (F_n \otimes I_6) \lim_{t \to \infty} \phi_{\tilde{x}}^e(t)(F_n \otimes I_6)^*x^e(0)
$$

$$
= (F_n \otimes I_6) \lim_{t \to \infty} \text{diag}(e^{At}, 0, \ldots, 0)(F_n \otimes I_6)^*x^e(0)
$$

$$
= ([1, 1, \ldots, 1]_n^T \otimes I_6) \lim_{t \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} (e^{At}x_i^e(0)) \right)
$$

(34)

That is

$$
x_{i,ss}^e(t) = \lim_{t \to \infty} \frac{1}{n} \sum_{i=1}^{n} (e^{At}x_i^e(0)), \quad \forall i = 1, 2, \ldots, n
$$

(35)

So the relative state of each satellite at steady states after the goal of formation control is achieved is

$$
x_{i,ss}(t) = \lim_{t \to \infty} (x_i^d(t) - x_i^{ss}(t))
$$

$$
= \lim_{t \to \infty} \left( x_i^d(t) - \frac{1}{n} \sum_{i=1}^{n} (e^{At}x_i^e(0)) \right),
$$

\forall i = 1, 2, \ldots, n

(36)

Using the above equation leads to

$$
\lim_{t \to \infty} \Delta x_{i,j}(t) = x_{j,ss}(t) - x_{i,ss}(t)
$$

$$
= \left( \lim_{t \to \infty} x_i^d(t) - x_i^{ss}(t) \right)
$$

$$
- \left( \lim_{t \to \infty} x_j^d(t) - x_j^{ss}(t) \right)
$$

$$
= \lim_{t \to \infty} (x_i^d(t) - x_j^d(t))
$$

$$
= \lim_{t \to \infty} \Delta x_{i,j}^d(t), \quad \forall i, j = 1, 2, \ldots, n.
$$

where $\Delta x_{i,j}^d(t)$ is the desired relative state of the $i$th satellite with respect to the $j$th satellite. As concluded from (37), the goal of the formation defined in (15) is achieved.

Now, the standing problem is now to design state feedback gain $K$ such that the system in (21) is asymptotically stable. The closed-loop system in (21) can be viewed as the following linear parameter uncertain system with state feedback control.

$$
\begin{cases}
\dot{\chi}(t) = A\chi(t) - \lambda_i B\nu(t); \\
\nu(t) = K\chi(t), \quad \forall i = 2, 3, \ldots, n
\end{cases}
$$

(38)

The optimal guaranteed cost control law to be introduced in Section 5 is chosen to design the above state feedback controller, because optimal guaranteed cost control law can stabilize the linear system with uncertain parameter while minimizing the LQR cost function. The choice of the weight matrices in the cost function is a tradeoff between the fuel consumption and the convergence rate.

**Remark 3.** The zero eigenvalue of $L_G$ can be interpreted as the uncontrollable mode of absolute motion of the satellite formation. Since $x_d$ is a known function and does not affect controllability, then dropping $x_d$ from (23) leads to the following:

$$
x(t) = \begin{bmatrix}
A - 2BK & BK & 0 & 0 & \cdots & 0 & BK \\
BK & A - 2BK & BK & 0 & \cdots & 0 & 0 \\
0 & BK & A - 2BK & BK & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
BK & 0 & 0 & 0 & \cdots & BK & A - 2BK
\end{bmatrix}_{6n \times 6n}
$$

(39)

Use the change of coordinates $\tilde{x} = Px$ such that

$$
\tilde{x}_1 = x_1, \tilde{x}_2 = x_2, \quad \tilde{x}_{n-1} = x_{n-1}, \quad \tilde{x}_n = \sum_{i=1}^{n} x_i
$$

(40)

which gives, with some manipulations,
As observed from (41), the following can be obtained

\[
\dot{x}(t) = \begin{bmatrix}
A-3BK & 0 & -BK & -BK - BK & \cdots & -BK - BK - BK
\end{bmatrix}
\begin{bmatrix}
BK & A-2BK & BK & 0 & \cdots & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & BK & A-2BK & BK & 0 & \cdots & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & BK & A-2BK & BK & 0 & \cdots & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & A
\end{bmatrix}x(t).
\]  

(41)

Therefore, the zero eigenvalue of \( L_G \) can be interpreted as the uncontrollable mode of absolute motion of the satellite formation.

5. Optimal Guaranteed Cost Control Design

5.1 Optimal Guaranteed Cost Control

Consider a class of linear uncertain systems described by the following state-space equation:

\[
\dot{x}(t) = (A + \Delta A)\bar{x}(t) + (\bar{B} + \Delta \bar{B})\bar{u}(t), \bar{x}(0) = \bar{x}_0 \tag{43}
\]

where \( A \) and \( \bar{B} \) are known constant real matrices of appropriate dimensions, \( \Delta A \) and \( \Delta \bar{B} \) are matrix-valued functions representing time-varying parameter uncertainties in the system model. The parameter uncertainties considered here are assumed to be norm-bounded and of the form:

\[
[\Delta A \ \Delta \bar{B}] = DF(t)[E_1 \ E_2]
\]  

(44)

where \( D, E_1 \) and \( E_2 \) are known constant real matrices of appropriate dimensions, which represent the structure of uncertainties and \( F(t) \) is an unknown matrix function with Lebesgue measurable elements and satisfies:

\[
F^T(t)F(t) \leq I
\]  

(45)

where \( I \) denotes the identity matrix of appropriate dimension.

Associated with the system in (43) is the cost function, where \( Q \) and \( R \) are given positive-definite symmetric matrices,

\[
J = \int_0^\infty [\bar{x}^T(t)Q\bar{x}(t) + \bar{u}^T(t)R\bar{u}(t)]dt
\]  

(46)

**Definition 1.** Consider the uncertain system in (43), if there exists a control law \( \bar{u}(t) \) and a positive scalar \( J^* \) such that for all admissible uncertainties, the closed-loop system is stable and the closed-loop value of the cost function in (46) satisfies \( J \leq J^* \), then \( J^* \) is said to be a guaranteed cost and \( \bar{u}(t) \) is said to be a guaranteed cost control law for the uncertain system in (43).

The objective of this section is to design a state-feedback guaranteed cost control law \( \bar{u}(t) = K\bar{x}(t) \) for the uncertain system in (43).

**Theorem 2** [19]. Given the system described (43) and the associated cost function in (46), if the following optimization problem:

\[
\min_{\epsilon, \ W, \ X, \ M} \text{Trace}(M)
\]  

subject to

\[
\begin{bmatrix}
\bar{A}X + \bar{B}W + (\bar{A}X + \bar{B}W)^T \epsilon DD^T (E_1X + E_1W)^T X \ W^T
\end{bmatrix} \leq 0
\]

\[
\begin{bmatrix}
M & I
I & X
\end{bmatrix} < 0
\]

(48)

has a solution \( \epsilon^*, \ W^*, \ X^* > 0, \ M^* > 0 \), then the state-feedback control law is the optimal guaranteed cost controller as follows:

\[
\bar{u}(t) = W^*(X^*)^{-1}\bar{x}(t)
\]  

(50)

and the associated guaranteed cost is,

\[
J \leq \text{trace}((X^*)^{-1}) = J^*
\]  

(51)

The above problem is a convex optimization problem subject to linear matrix inequalities constraints and can be solved by using LMI toolbox.

5.2 Guaranteed Cost Controller Design

Now the problem is making use of the optimal guaranteed cost control to design a state-feedback controller that can asymptotically stabilize the system in (21) for each nonzero eigenvalue \( \lambda_i \) of \( L_G \). The system in (21) can be expressed as the following linear norm-bounded uncertain system:

\[
\dot{x}(t) = A\chi(t) + (\bar{B} + \Delta \bar{B})\nu(t)
\]  

(52)

where
The satellite motion under cyclic formation control is shown in Fig. 3. As illustrated from this figure, the geometry centre of the circular formation deviates from the reference orbit, because only initial relative state of each satellite, but not the reference orbit, contribute to the relative state of each satellite at steady state, as can be seen from Theorem 1. Figure 4 shows the formation error $\epsilon_{i,j} = \Delta x^e_{i,j}$ between satellite $i$ and satellite $j$. As observed from this figure, the convergence time of formation control is less than 1000 s. Figure 5 shows the control input of each satellite.

The choice of the state feedback control gain, which is subject to the available maximum thrust, is a tradeoff between fuel consumption and convergence rate. As an illustration, the following case is considered. Suppose the convergence rate needs to be increased, LQR performance weight matrices in (46) can be decreased to $Q = I_6$ and $R = 10^3 I_3$. The state feedback gain obtained by using optimal guaranteed cost control is given by,

$$ a = 7350 \text{ km}, \quad e = 0.001, \quad i = 45^0, \quad \omega = 30^0, \quad \Omega = 45^0, \quad f = 0^0 $$

The minimum and maximum nonzero eigenvalues of $L_G$ are $\lambda_{\text{min}} = 1$, $\lambda_{\text{max}} = 4$, respectively, since there are six satellites in the formation in our test scenario. To save fuel consumption, LQR performance weight matrices in (46) are chosen as $Q = I_6$ and $R = 10^3 I_3$. The state feedback gain obtained by using optimal guaranteed cost control is given by,

$$ K = \begin{bmatrix}
8.202 \times 10^{-3} & 3.373 \times 10^{-5} & 4.805 \times 10^{-5} & -8.005 \times 10^{-6} & 0 & 0 \\
4.805 \times 10^{-5} & 8.041 \times 10^{-6} & 7.834 \times 10^{-3} & 3.059 \times 10^{-5} & 0 & 0 \\
0 & 0 & 0 & 7.828 \times 10^{-3} & 3.064 \times 10^{-5} & 0
\end{bmatrix} $$

The minimum and maximum nonzero eigenvalues of $\Delta \lambda = \lambda_{\text{max}} - \lambda_{\text{min}}$ among $\lambda_i = 2 - 2 \cos(2\pi(i-1)/n)$, $\forall \ i = 2, 3, \ldots, n$. Then the state-feedback control gain for the norm-bounded uncertain system in (52) can be calculated using Theorem 2.

6. Numerical Simulations

In this section, simulations using STK and Matlab demonstrate the performance of the proposed cyclic formation control for satellite formation. STK is a worldwide software package that allows engineers and scientists to design and develop complex dynamic simulations of Earth-orbiting satellites. In the process of simulation, the control inputs are calculated in Matlab using data transmitted from STK via STK Connect. The motion of each satellite is propagated through STK Real-time propagator with J2, J3 and J4 perturbations. For illustration, the scenario of keeping a circular formation is considered, in which six satellites are evenly spaced in a circle with radius of 1000 m. The desired initial relative position and velocity of each satellite in local LVLH frame to form a circular formation are given by (53) [21]. The desired relative position and velocity of each satellite are then obtained by solving CW equations with desired initial conditions.

$$ \begin{align*}
\theta_1 &= 0^0, \quad \theta_2 = 60^0, \quad \theta_3 = 120^0, \quad \theta_4 = 180^0, \quad \theta_5 = 240^0, \quad \theta_6 = 300^0, \\
x_{i,0} &= (r_c/2) \cos \theta_i, \quad \dot{x}_{i,0} = (r_c/2 \omega_c) \sin \theta_i, \\
y_{i,0} &= 2 \dot{x}_{i,0} / \omega, \quad \dot{y}_{i,0} = -2 \omega_c x_{i,0}, \\
z_{i,0} &= \sqrt{3} x_{i,0}, \quad \dot{z}_{i,0} = \sqrt{3} \dot{x}_{i,0}, \quad \forall \ i = 1, 2, \ldots, 6
\end{align*} \tag{53} $$

The actual initial relative positions and velocities for the six satellites in our test scenario have initial errors as compared to the desired relative state. The reference orbit is characterized by the following orbit elements:

$$ a = 7350 \text{ km}, \quad e = 0.001, \quad i = 45^0, \quad \omega = 30^0, \quad \Omega = 45^0, \quad f = 0^0 \tag{54} $$

The choice of the state feedback control gain, which is subject to the available maximum thrust, is a tradeoff between fuel consumption and convergence rate. As an illustration, the following case is considered. Suppose the convergence rate needs to be increased, LQR performance weight matrices in (46) can be decreased to $Q = I_6$ and keep $R = 10^3 I_3$ constant. Then the state feedback gain calculated by the optimal guaranteed cost control as given below, is larger in the sense of matrix norm than the previous state feedback gain.
$K = \begin{bmatrix}
4.479 \times 10^{-2} & 1.003 \times 10^{-3} & 1.500 \times 10^{-6} & -4.477 \times 10^{-5} & 0 & 0 \\
1.500 \times 10^{-6} & 4.477 \times 10^{-5} & 4.473 \times 10^{-2} & 9.997 \times 10^{-4} & 0 & 0 \\
0 & 0 & 0 & 0 & 4.473 \times 10^{-2} & 9.998 \times 10^{-4}
\end{bmatrix}$  \hspace{1cm} (56)

Figure 3. Satellite motion under cyclic formation control and simulated with STK and Matlab.

Figure 4. Formation errors among satellites under cyclic formation control.
Figure 5. Control input to each satellite under cyclic formation control.

Figure 6. Formation errors among satellites under cyclic formation control with larger state feedback gain.
In the case of larger state feedback gain, the formation control errors are shown in Fig. 6. As observed from this figure, the convergence time of the formation control is limited within 200 s, which is much less than that of the previous controller.

7. Conclusions

An efficient cyclic formation control method for satellite formation using only local relative measurements is proposed in this paper. Compared to the work of Gurfil and Mishne [18], this proposed approach achieves both bounded formation, in which the energy of each satellite is matched, and precise control of the relative positions and velocities between each of the satellites. In the proposed methodology, each satellite controls itself by using only the local relative measurements from itself to its two neighbouring satellites. The goal of formation control for $n$ satellites is achieved if the state-feedback controller stabilizes relative dynamics, individualized by a specific scalar. The scalar takes values according to nonzero eigenvalues of a circulant matrix $L_G$ representing the topology of relative measurements among the satellites. The proposed cyclic formation control architecture is completely decentralized and scalable. Only local relative measurements are needed and no infrastructure for full collaboration and communication among the satellites is required in the proposed controller.

Acknowledgments


References


Biographies

Baolin Wu received the B.S. and the M.S. degrees from the Harbin Institute of Technology, Harbin, People’s Republic of China, in 2003 and 2005, respectively. He is now completing the Ph.D. degree in the Aerospace Electronics Lab, School of Electrical and Electronics Engineering, Nanyang Technological University, Singapore. His current research interests are in the area of satellite formation control, trajectory optimization, and attitude control.
Danwei Wang received the B.E. degree in electrical engineering from the South China University of Technology, Guangzhou, China, in 1982, and the M.S.E. and Ph.D. degrees in electrical engineering from the University of Michigan, Ann Arbor, in 1984 and 1989, respectively. Since 1989, he has been with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, where he is currently a Professor and the Head of the Division of Control and Instrumentation. He is an Associate Editor for the International Journal of Humanoid Robotics. He is the author or coauthor of more than 200 technical articles in the areas of iterative learning control, repetitive control, robust control, and adaptive control systems, as well as manipulator/mobile robot dynamics, path planning, and control. His current research interests include robotics, control theory, and applications. Prof. Wang has been a General Chairman, a Technical Chairman, and has held various positions in international conferences, such as the International Conference on Control, Automation, Robotics, and Vision (ICARCVs) and the International Conference on Intelligent Robots and Systems (IROS). He has also been an Associate Editor of the Conference Editorial Board, IEEE Control Systems Society from 1998 to 2005. He was a recipient of the Alexander von Humboldt Fellowship, Germany.

Eng Kee Poh graduated from the National University of Singapore with a Bachelor of Engineering in 1986. He received the M.S. and the Ph.D. degrees from the University of Michigan, USA, in 1990 and 1993, respectively. Since then, he has assumed the appointments of Laboratory Head, Chief Engineer, Program Director, and Program Manager at DSO National Laboratories from 1994 to 2007. Presently, he is an Program Manager of Guidance, Navigation and Control program at DSO National Laboratories where he also holds the title of Distinguished Member of Technical Staff. Concurrently, he is also an adjunct Associate Professor at the School of Electrical and Electronic Engineering, NTU where he conducts postgraduate courses in computational studies and control and also supervises Ph.D. students in the areas of control and signal processing.