MAXIMUM LIKELIHOOD DETECTION AND ESTIMATION OF
BERNOULLI GENERALIZED GAUSSIAN PROCESSES WITH
NON-GAUSSIAN COLORED NOISE

Akram BELGHITH, Christophe COLLET

c.collet@unistra.fr

ABSTRACT
In this paper we address the problem of restoration of the wavelet coefficients related to crackle assumed to be a pulse respiratory signal with a non-gaussian colored noise. This task, requiring multivariate probability density computations for the data likelihood term, often faces with the lack of analytical multidimensional expressions in the non-gaussian case. Thus, multidimensional Gaussian distribution is usually used for its simplicity, even if Gaussian assumption is not always verified. In this work, we propose a new approach based on copula theory to compute multivariate generalized Gaussian marginals to deal with the non-gaussianity of the wavelet coefficients of the colored added noise for restoration of the respiratory signals.

KEY WORDS
Deconvolution, respiratory signal, Generalized Gaussian distribution, Copulas theory, Biomedical.

1 Introduction
Since the invention of the stethoscope, respiratory sounds acoustic analysis has been used to evaluate and diagnose patients with lung diseases. Nevertheless, this method has a high degree of subjectivity relative to the specialist. During the last decade, sound signal digitization and processing techniques have been developed [1, 2] contributing to make more objective the method by means of quantitative data. Respiratory sounds can be divided into normal and abnormal categories according to their acoustic properties. Crackles are adventitious sounds with a very short duration (< 20ms) [3]. They give information about lung airways activity, correlated with their obstruction level. Thus, these sounds are useful to detect respiratory chronic obstructive diseases and study their evolution such as the Chronic Obstructive Pulmonary Disease COPD.

COPD is a disorder characterized by reduced maximum expiratory flow and slow forced emptying of the lungs; features which do not change markedly over several months [4]. Most of the airflow limitation is slowly progressive but irreversible. The airflow limitation is due to varying combinations of airway disease and emphysema, the relative contribution of the two processes is difficult to define in vivo. The airway component consists mainly of decreased luminal diameters due to various combinations of increased wall thickening, increased intraluminal mucus, and changes in the lining fluid of the small airways [5]. The most difficult diagnostic problem is to distinguish COPD from the persistent airflow limitation of other respiratory diseases such as a chronic asthma in older subjects. The detection of some physical sings may help in COPD diagnose. Among the classical physical signs, presence of the crackles are useful indicators of airflow limitation.

Our aim is to be able to count the crackles and to estimate their amplitudes and their appearance density to quantify the gravity of the COPD. In our case, the crackle is assumed to be an impulse signal because of its duration. With such condition, the only characteristics of the signal attributable to the pathology signature are its position and its amplitude [6]. The impulse deconvolution scheme offers the possibility of estimating simultaneously the position and the amplitude of every impulse.

In the literature, there are several impulse deconvolution methods such as the adaptive filter [7] which is a satisfactory method only when the impulses are well separated. A second family of impulse deconvolution methods uses a Bayesian framework [8] to model the impulses by a Bernoulli-gaussian (BG) process. The interest of such approach is in its capacity to take into account the information brought by the measures and the a priori information that we have. The adequacy between the estimated parameters and the model is described thanks to a likelihood function which must be maximized.

A simple study shows that the wavelet coefficients of the crackles and the normal respiratory sound do not follow a gaussian distribution. This gaussianity is a basic hypothesis of the BG deconvolution. Several studies show the effectivity of the GG distribution for the modelisation of the wavelet signal coefficients [9, 10]. Therefore, we choose the GG density to modelize the wavelet coefficients of the crackles and the normal respiratory sound. Besides,
the hypothesis of a white gaussian noise is not verified in practice on the acquired lung signals. In fact, both sounds, normal respiratory sound and respiratory sounds with crackles, are the result of the air circulation in the lung which provokes the signals transformations. These transformations can be modeled thanks to two FIR filters [11]. Hence, the noise at the output is not any more a white noise but a colored noise.

In order to obtain an adequate solution, the non-gaussianity nature of the wavelet coefficients of the colored added noise and the distribution of the crackle wavelet coefficients must be accurately modeled. In a stochastic framework, this can be done by modeling the pulse train as a Bernoulli-Generalized-Gaussian (BGG) process and using the theory of copula to compute multivariate GG marginals. By this way, we may be able to deal with the non-gaussianity of the wavelet coefficients of the colored added noise and then use a Bayesian estimator for the restoration scheme.

In section (2) we present our modeling assumptions, where as section (3) describes the copula theory and presents the multivariate GG marginal. In section (4) we derive the Single Most Likely Replacement (SMLR) detector and in section (5) we present how the amplitude estimation is processed. In (6) we present deconvolution results on a synthetic signal and then on different biological signals.

2 Problem Formulation

We begin with the discrete-time convolution model:

\[ z(n) = \sum_{i=1}^{m} h(i) r(n-i) + b(n) \quad (1) \]

where \( z(n) \) is observed data, \( h(i), 1 \leq i \leq m \) is the impulse response of the linear system, \( r(n) \) is the impulsive signal to be estimated and \( b(n) \) stands for the noise. We can easily adopt the matrix formulation through which the input-output equation can easier be written in matrix form:

\[ z = Hr + b \quad (2) \]

where vectors \( z, r, n \) contain the samples of the observed signal, of the input signal and of the observed noise, respectively. \((N \times N)\) matrix \( H \) is made up of shifted samples of the impulse response of the linear system.

Here, the observation noise is assumed to be a zero-mean GG process with covariance matrix \( \Sigma_b \) and shape parameter \( \alpha_b \) where the GG marginal density is given by [12] :

\[ f(z; \alpha, \sigma, \mu) = \frac{\eta(\alpha)\alpha}{2\Gamma(1/\alpha)} \exp\left[-\eta(\alpha)|z-\mu|^{\alpha}\right] \quad (3) \]

where \( \eta(\alpha) = \left[ \frac{\Gamma(1/\alpha)}{\sigma\Gamma(1/\alpha)} \right]^{\frac{1}{\alpha}} \), \( \Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \exp^{-t} \, dt \) and \( \mu, \sigma, \alpha \) the mean, standard deviation and shape parameter. The impulsive input sequence \( r(n) \) is statistically modeled as a zero-mean BGG sequence; i.e., event times are independently distributed with probability \( \lambda \) (Bernoulli sequence), and event amplitudes are GG distributed with mean zero, standard deviation \( \sigma \) and \( \alpha \) for shape parameter. To conceptually separate the amplitude estimation problem from the event detection one, we find it convenient to express sequence \( r(n) \) using the product model:

\[ r(n) = a(n)q(n) \quad (4) \]

where \( a(n) \) is GG function whose parameters are \( \sigma \) and \( \alpha \) \((Pr = 0)\), and \( q(k) \) is a Bernoulli sequence, for which:

\[ Pr\{q(k)\} = \begin{cases} 1 - \lambda & q(k) = 0 \\ \lambda & q(k) = 1 \end{cases} \quad (5) \]

Event detection consists in finding maximum likelihood estimates \( \hat{q}(k), k = 1, \cdots, N \) and amplitude estimation consists in finding maximum likelihood estimates \( \hat{a}(k), k = 1, \cdots, N \).

In this paper, we assume that the impulse response of the linear system \( h \) is a GG filter. In fact, Fig. 1 shows the wavelet coefficients of the crackle which is fitted with a GG filter presented with dotted line. As on can see, the GG filter may be used as a good fitter for the wavelet coefficients of th crackle.

![Figure 1. Wavelet coefficients of a crackle fitted with a GG filter (zoom).](image)

3 Multivariate GG Marginal

The deconvolution problem with matrix formulation requires that the probability density function be in multivariate form. In general, the computation of multivariate distribution is not trivial in the non-Gaussian case. Then
likelihood computation corresponds to the product of the marginals under independence assumption:

\[ \forall \mathbf{z} = (z(1), \ldots, z(N)) \in \mathbb{R}^N \quad f(\mathbf{z}) = \prod_{n=1}^{N} f(z(n)) \quad (6) \]

Since such hypothesis is often not verified, a decorrelation step may be operated using ICA [13] before using the product. However this problem can be solved more gracefully with the copula theory. The basis of this theory is the Sklar Theorem [14] which asserts the existence of a function C, called copula and defined on \([0, 1]^N\), binding the joint cumulative distribution function \(F(y^1, \ldots, y^N)\) to the marginal cumulative distribution functions \(F^{[1]}(y^1), \ldots, F^{[N]}(y^N)\) (where \(y = (y^1, \ldots, y^N)\) contains the samples of observed signal in different bands), as follows:

\[
F(\mathbf{y}) = C(F^{[1]}(y^1), \ldots, F^{[N]}(y^N)) \quad (7)
\]

If the marginals \(F^{[1]}, \ldots, F^{[N]}\) are continuous, then \(C\) is unique. Moreover, if \(C\) is differentiable it is possible to define a copula density as [14]:

\[
f(y^1, \ldots, y^N) = f^{[1]}(y^1) \times \cdots \times f^{[N]}(y^N) \times \cdots
c(F^{[1]}(y^1), \ldots, F^{[N]}(y^N)) \quad (8)
\]

where \(f^{[j]}(y^j)\) is the probability density function corresponding to \(F^{[j]}(y^j)\) and \(c = \partial C / \partial F^{[1]} \cdots \partial F^{[N]}\) is the copula density.

Several studies show the effectivity of the Gaussian copula to handle dependence [15, 16]. For multivariate Gaussian copula \(C_G\) is given by [17, 18]. \(\forall \mathbf{y} = (y^1, \ldots, y^N) \in \mathbb{R}^N:\)

\[
c_G(\mathbf{y}, \mathbf{R}) = |\mathbf{R}|^{-\frac{1}{2}} \exp \left[ -\frac{\mathbf{y}^T (\mathbf{R}^{-1} - I) \mathbf{y}}{2} \right] \quad (9)
\]

where \(\mathbf{y} = (\Phi^{-1}(y^1), \ldots, \Phi^{-1}(y^N))^T\) with \(\Phi(\cdot)\) the standard Gaussian cumulative distribution, \(\mathbf{R}\) is the inter-band correlation matrix and \(I\) the \(N \times N\) identity matrix. If we consider that the elements of the vector of observations have the same probability density function, we can calculate the expression of the Gaussian copula using the covariance matrix as follows:

\[
\forall \mathbf{z} = (z(1), \ldots, z(N)) \in \mathbb{R}^N:
\]

\[
c_G(\mathbf{z}, \Sigma) = \frac{1}{\det(\Sigma)} \exp \left[ -\frac{\mathbf{z}^T (\Sigma^{-1} - \text{diag}(\Sigma^{-1})) \mathbf{z}}{2} \right] \quad (10)
\]

where \(\Sigma\) is the covariance matrix.

To model non-Gaussian multivariate densities, we use Eq. 8 with a Gaussian copula density Eq. 10 and GG marginal densities Eq. 3.

Finally the expression of the multivariate GG density is given by:

\[
f(z; \alpha, \sigma, \mu, \Sigma) = \prod_{i=1}^{N} f(z(i); \alpha, \sigma, \mu) c_G(z, \Sigma) \quad (11)
\]

4 Single Most Likely Replacement (SMLR) Detection

The restoration of the pulse process requires two operations: detection of position variables \(q(k)\) and estimation of amplitude variables \(a(k)\). The derivation of a MAP estimator of \((q, a)\) consists of maximizing the following joint likelihood:

\[
Pr\{ (q, a) / z \} \propto Pr(z / a, q) Pr(a / q) Pr(q) \quad (12)
\]

To avoid a high number of false detections, we adopt a sequential approach where detection is performed alternatively through maximization of the posterior marginal likelihood of \(q\) then that of amplitudes \(r\) [19]. The posterior marginal likelihood of \(q\) is given by:

\[
Pr(q / z) \propto Pr(z / q) Pr(q) \quad (13)
\]

To compute \(Pr(z / q)\), we assume that \(r\) and \(b\) given \(q\) follow GG distributions, but their sums neither follow a GG distribution nor has an analytical expression [20]. To deal with this problem, we will use a numerical integration to obtain an approximation of the exact pdf. In fact, we know that a convolution of two or more independent variables is the probability distribution of the sum of those variables. This approximation is often used when the analytical expression of the convolution product is unknown [21]. Note that this assumption may be justified since we try to maximize the likelihood, so it is not necessary to find the exact value of \(Pr(z / q)\).

Since the \(q(k)\) are independently distributed, we show from (5) that:

\[
Pr(q) = \prod_{i=1}^{N} Pr(q(i)) = \lambda_{N_q} (1 - \lambda)^{N - N_q} \quad (14)
\]

where \(N_q\) is the number of nonzero elements in \(q\). We can see that the elements of \(q\) have nonlinear interactions in \(Pr(q / z)\). So, the only way to maximize the expression is to compute the likelihood given the values of the entire \(q\)-sequences \((2^N\) evaluations)[22].

The SMLR detector can be interpreted in this framework. In fact, the SMLR algorithm is an iterative search algorithm that compares the ward likelihood of two sequences: the reference sequence \(q_r\) to the test sequences,
In each iteration $q_t$ differs from the $q_i$ only by one sample. The selection strategy consists of choosing as the next current sequence the one that maximizes the criterion given by:

$$C_{r,t} = \log \left( \frac{Pr(q_t/z)}{Pr(q_i/z)} \right) \geq 0$$

(15)

The expression of the $q_i$ in the $i$-th iteration giving the previous sequence reference $q_i$ is given by:

$$q_t(k) = \begin{cases} 1 - q_i(i) & i = k \\ q_i(i) & \text{otherwise} \end{cases}$$

(16)

Finally, the new reference sequence in the $i$-th iteration is given by:

$$q_r = \begin{cases} q_t & \text{if } C_{r,t} < 0 \\ q_i & \text{if } C_{r,t} > 0 \end{cases}$$

(17)

The SMLR algorithm can be summarized as follows:

Algorithm 1 SMLR

**Initialization** of $q_r$ : Generate a $N$ Bernoulli sequence with probability $\lambda$

for $i = 1$ to $N$ do

Generate $q_r$ as (16)

Verify the criterion (15)

Update the reference sequence $q_r$ (17)

end for

5 Estimating Amplitudes

Now, we consider the problem of amplitude estimation of the impulsive input sequence $r$. The derivation of a MAP estimator of $r$ consists in maximizing the following likelihood:

$$\hat{r} = \arg \max_r [Pr(r/z, \hat{q})]$$

(18)

where

$$Pr(r/z, \hat{q}) \propto Pr(z/r, \hat{q}) Pr(r/\hat{q})$$

(19)

Since $r$ with respect to $\hat{q}$ are independently distributed, we show from (4) that:

$$Pr(r/\hat{q}) = \prod_{i=1}^{N} \left[ \frac{\eta(\alpha_r)\alpha_r}{2\Gamma(\frac{1}{\alpha_r})} \right] \exp \left[ - (\eta(\alpha_r) |r(i)|)^{\alpha_r} \right]$$

(20)

where $\sigma_r$ the standard deviation and $\alpha_r$ the shape parameter of $r$. From the input-output equation (2) we prove that:

$$Pr(z/r, \hat{q}) = \prod_{i=1}^{N} \left[ \frac{\eta(\alpha_h)\alpha_h}{2\Gamma(\frac{1}{\alpha_h})} \right] \exp \left[ - (\eta(\alpha_h) |z(i) - (Hr)(i)|)^{\alpha_h} \right]$$

$$\times c_G(z, \Sigma_h)$$

(21)

In fact:

$$E_{z/r, \hat{q}}[z] = E_{z/r, \hat{q}}[Hr + b] = Hr$$

(22)

and

$$E_{z/r, \hat{q}}[zz^T] = E_{z/r, \hat{q}}[(Hr + b)(Hr + b)^T] = \Sigma_h$$

(23)

Given Eq. 20 and 21, we solve the optimization task described in Eq.18.

6 Deconvolution Results

To validate our method of deconvolution we consider a synthetic signal designed to fit our modeling assumptions. For this, we generate a BGG sequence ($\lambda = 0.05$, $N = 500$, $N_q = 20$, $\sigma_q = 0.1$, $\alpha_q = 1$). Then, this sequence is convoluted with a GG filter ($\alpha = 0.5$, $\sigma = 0.3$) shown in Fig. 2 for which a GG colored noise ($\alpha_h = 1.4$) is added to produce the synthetic signal shown in Fig. 3.

Figure 2. Impulse response of the linear system used for the simulation

Figure 3. The synthetic signal used for the simulation (signal-to-noise ratio of 9 dB).
Since we deal with the supervised deconvolution, we assume the knowledge of the hyperparameters of the BGG processes and the noise. Theoretically, the results of the deconvolution depend on these hyperparameters which can be estimated with the method presented in [12]. For the covariance matrix of the colored noise, a robust estimator can be derived from the wavelet coefficients using the estimator described in [23]. To have an estimate of $\lambda$, we can use the results produced by the Wiener filter. Note that a fully Bayesian approach employing Monte Carlo Markov chain (MCMC) algorithm [24] can also be adopted but its computational cost may become prohibitive. To prove the robustness of our MAP estimator to the deviation of these parameters from their presumed values we apply the proposed method to the synthetic signal. The detected estimates with different values of some hyperparameters are summarized in Table 1. As one can see, our algorithm is sufficiently robust to errors in the estimated parameters.

In order to emphasize the benefit of the proposed approach and particularly the use of the GG to modelize the wavelet coefficients of the colored added noise, four different detectors were applied to the synthetic signal presented in Fig. 2: a Viterbi detector (M 1) [25], a recursive detector (M 2) [26], SMLR Gaussian detector, i.e., the observation noise is assumed to be a Gaussian white noise (M 3) and our Estimator method BGG-SMLR (M 4). The results of deconvolution are summarized in Table 2. As one can see, our SMLR detector performed the best.

<table>
<thead>
<tr>
<th>hyperparameters</th>
<th>$\alpha_r$</th>
<th>$\alpha_b$</th>
<th>$\sigma^2_r$</th>
<th>$\lambda$</th>
<th>missed detections</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.4</td>
<td>0.1</td>
<td>0.05</td>
<td>3/20</td>
<td></td>
</tr>
<tr>
<td>0.91</td>
<td>1.4</td>
<td>0.1</td>
<td>0.05</td>
<td>3/20</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.48</td>
<td>0.1</td>
<td>0.05</td>
<td>3/20</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.4</td>
<td>0.19</td>
<td>0.05</td>
<td>4/20</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.4</td>
<td>0.1</td>
<td>0.09</td>
<td>3/20</td>
<td></td>
</tr>
</tbody>
</table>

Table 2

Performances of different detection methods

<table>
<thead>
<tr>
<th>missed detections</th>
<th>M 1</th>
<th>M 2</th>
<th>M 3</th>
<th>M 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8/20</td>
<td>7/20</td>
<td>5/20</td>
<td>3/20</td>
</tr>
</tbody>
</table>

Now, we consider the problem of the real signal deconvolution. Our method was tested on many biologic signals which correspond to the COPD sound (the respiratory sounds produced by the persons are affected with COPD). The Fig. 5 presents a raw biologic signal in the time domain. The overview diagram of the treatment chain is presented in Fig. 6. Firstly, we calculate the wavelet packet transformation of the signal and we detect respiratory phases [27]. Then, we apply our SMLR detector to the COPD wavelet packet coefficients. We obtain the detected estimates presented in Fig. 7. The signal presents the COPD wavelet packet coefficients and the circle marks describe the corresponding estimates. By this way, we should be able to detect and quantify the development of the COPD disease.

As one would suspect, the missed detections correspond to weak events which are buried in the noise.

![Figure 4. SMLR detector estimates](image)

![Figure 5. COPD sound (time in axis x)](image)
Wavelet packet selection

↓

Respiratory phase detection

↓

Deconvolution

↓

Detection and quantification of COPD

Figure 6. Overview diagram of the treatment chain.

Figure 7. The COPD wavelet packet coefficients and the corresponding estimates (time in axis x)

7 Conclusion

This paper addressed the problem of the deconvolution of a BGG process with the presence of non Gaussian colored noise, in our case a GG colored noise, by the maximization of the likelihood function faces with the lack of an analytical multidimensional density expression in the non-gaussian case. We proposed a new approach based on copula theory to compute multivariate GG distributions to deal with the non-gaussianity of the wavelet coefficients of the colored added noise. We currently work now on the validation of our method on a raw biological signal data base in collaboration with physicians.

Acknowledgements

We would like to thank ANR ASAP for funding, and to Hopitaux Universitaires de Strasbourg (Pr E. Andrs, service de medicine interne, clinique mdical B) for the respiratory data available on http://www.websound.fr.

References


