A PROPOSED WARPED WIGNER-VILLE TIME FREQUENCY DISTRIBUTION APPLIED TO DOPPLER BLOOD FLOW MEASUREMENT

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ABSTRACT
One of the main goals in ultrasonic Doppler blood flow measurement is the estimation of the mean velocity. The Doppler signal’s instantaneous frequency has traditionally been used to estimate the mean velocity. In this work, a non-uniform discrete time frequency distribution is proposed: the warped discrete Wigner-Ville distribution (WTFD_WV). The proposed procedure estimates the instantaneous frequency by concentrating the frequency resolution around the instantaneous frequency to adjust a parabola on the main spectral lobe. The parabola’s maximum is then located over the instantaneous frequency. As a result, a better precision is obtained in the spectral estimation by using a WTFD_WV for noisy signals when compared to other methods such as the Discrete Wigner-Ville Time Frequency Distribution (DTFD_WV) with the instantaneous frequency calculated as the centroid of the spectrum. It is observed that the WTFD_WV acts as a band-pass filter around the instantaneous frequency. Additionally, this paper proposes a generalisation of the WTFD_WV as a Warped Discrete Time Frequency Distribution Class.

KEY WORDS

1. Introduction
It is known that the blood flow mean velocity through a vessel’s cross section is proportional to the instantaneous frequency of the Doppler ultrasonic signal. This work is focused on the accurate computation of the instantaneous frequency of a signal (Carotid Artery simulated signal) in the presence of noise.
A classic method to estimate the instantaneous frequency of a signal includes the computation of its spectrogram using a Short Time Fourier Transform (STFT). However, it assumes that the analysed signal is stationary and it compromises its temporal and frequency resolution. An alternative method is to use the Cohen Class Time Frequency Distributions that overcomes the stationary assumption. However, in some cases, it is desirable to increase only its frequency resolution around certain frequency of interest, for example, around the instantaneous frequency. That can be achieved if the length of the analysed discrete signal is increased but it also increases notably the computational cost. On the other hand, non-uniform discrete Fourier transforms are available such as the Warped Discrete Fourier Transform, which is able to achieve that task without having to increase the length of the analysed discrete signal. The aim of the work presented in this paper is to incorporate a warped frequency scale (non-uniform) to the Cohen class time-frequency distributions. We focus on the development of the warped discrete Wigner-Ville time-frequency distribution, although the procedure can be easily extended to other distributions.

2. Time Frequency Distributions of the Cohen Class
The time frequency distributions of the Cohen class (TFD) are defined as follows [1]. Let $\phi(\theta, \tau)$ be the distribution kernel. That kernel determines the distribution and its characteristics of temporal and frequency resolution. Let also $\psi(t, \tau)$ be the Fourier transform $F: \theta \rightarrow t$ of the distribution kernel. Then, let

$$R_i(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(t - \mu, \tau) \chi(\mu + \frac{i}{2} \tau) \chi^*(\mu - \frac{i}{2} \tau) d\mu$$

be the generalised local auto-correlation function, where $\chi(t)$ is a complex signal. It is important to point out that in this work, analytic signals will be used. Finally, let

$$TFD(t, \omega) = \int_{-\infty}^{\infty} R_i(\tau)e^{-i\omega\tau} d\tau$$

be the time frequency distributions, which are defined as the Fourier transform $F: \tau \rightarrow \omega$ of the generalised local auto-correlation function, where $t$ is the time variable and $\omega$ is the (angular) frequency variable.

3. Wigner Ville Time Frequency Distribution
The Wigner Ville time frequency distribution (TFD_WV) belongs to the Cohen class. Its kernel is the unit
\[ \phi(\theta, \tau) = 1, \text{ while its generalised local auto-correlation function is:} \]
\[ R_i(\tau) = x(t + \frac{1}{2}\tau)x^*(t - \frac{1}{2}\tau) \]  
\[ \text{(3)} \]

Then, the distribution is defined by:
\[ TFD_{WV}(t, \omega) = \int_{-\infty}^{\infty} x(t + \frac{1}{2}\tau)x^*(t - \frac{1}{2}\tau)e^{-j\omega\tau}d\tau \]  
\[ \text{(4)} \]

and the direct discretization of expression (4) constitutes the discrete distribution (DTFD_{WV}), that is:
\[ DTFD_{WV}(n, k) = 2 \sum_{r=-M+1}^{M-1} W(r)W^*(-r)x(n+r)x^*(n-r)e^{-j2\pi kr} \]  
\[ \text{(5)} \]

where \( n = -M + 1, \ldots, M - 1 \) is the time discrete variable, \( k = 0, \ldots, M - 1 \) is the frequency discrete variable, \( x(n) \) with \( n = -M + 1, \ldots, M - 1 \) is a discrete complex signal whose length is \( 2M - 1 \), and \( W(n) \) is a sampling window. In this work, a Hanning sampling window will be used.

Now, the discrete Wigner Ville time frequency distribution with periodic extension (PTFD_{WV}) is stated [2]. Although this procedure is illustrated only for the Wigner Ville distribution, its generalisation is direct. The procedure essentially consists on the following. First the discrete distribution (5) is valued at \( n = 0 \):
\[ DTFD_{WV}(0, k) = 2 \sum_{r=-M+1}^{M-1} W(r)W^*(-r)x(0)x^*(-0)e^{-j2\pi kr} \]  
\[ \text{(6)} \]

Second, the generalised local auto-correlation function is identified:
\[ R_i(\tau) = W(\tau)W^*(-\tau)x(\tau)x^*(-\tau) \]  
\[ \text{(7)} \]

where \( \tau = -M + 1, \ldots, M - 1 \), with a length \( L = 2M - 1 \). Now, the function \( \tilde{R}_i(\tau) \) is constructed, where \( \tau = 0, \ldots, L \), with a length \( \tilde{L} = L + 1 = 2M \), as follows:
\[ \tilde{R}_i(\tau) = \begin{cases} 
R_i(\tau) & 0 \leq \tau \leq M - 1 \\
0 & \tau = M \\
R_i(\tau - 2N) & M + 1 \leq \tau \leq L 
\end{cases} \]  
\[ \text{(8)} \]

which constitutes the periodic extension of \( R_i(\tau) \).

Finally, note that (6) can be written as:
\[ DTFD_{WV}(0, k) = 2 \sum_{r=-M+1}^{M-1} \tilde{R}_i(\tau)e^{-j2\pi kr} = 2 \sum_{r=-M+1}^{M-1} R_i(\tau)e^{-j2\pi kr} \]  
\[ \text{(9)} \]

and the latter expression can be written as:
\[ DTFD_{WV}(0, k) = \sum_{i=0}^{\tilde{L} - 1} \tilde{R}_i(\tau)e^{-j2\pi kr} = \sum_{i=0}^{\tilde{L} - 1} R_i(\tau)e^{-j2\pi kr} \]  
\[ \text{(10)} \]

At this point, the following scaling in the frequency axis is carried out. It consists on reducing by half the frequency resolution. The result is denominated a discrete Wigner Ville time frequency distribution with periodic extension (PTFD_{WV}):
\[ PTFD_{WV}(0, k) = 2 \sum_{i=0}^{\tilde{L} - 1} \tilde{R}_i(\tau)e^{-j2\pi kr} \]  
\[ \text{(11)} \]

4. Warped Discrete Fourier Transform

The main characteristic of the warped discrete Fourier transform (WDFT) [3][4][5] is that it can concentrate the frequency resolution around a frequency of interest, since it possesses a non-uniform frequency resolution. Contrary to the conventional discrete Fourier transform (DFT) that possesses a uniform frequency resolution.

The warped discrete Fourier transform with a first order all-pass filter is defined as:
\[ WDFT(k) = \sum_{n=0}^{N-1} x(n) \left[ \alpha^* + e^{-\frac{j2\pi kn}{N}} \right] ^\frac{1}{2} \]  
\[ \text{(12)} \]

where \( \alpha = |\alpha|\exp(j\varphi) \) is a complex parameter that determines the warped frequency scale, \( x(n) \) with \( n = 0, \ldots, N - 1 \) is a complex discrete signal with length \( N \), and \( k = 0, \ldots, N - 1 \) is an index related to the discrete frequency.

The warped frequency scale mapping is given by:
\[ \Omega_w = \Omega + 2 \arctan \left( \frac{|\alpha|\sin(\varphi - \Omega)}{1 + |\alpha|\cos(\varphi - \Omega)} \right) \]  
\[ \text{(13)} \]

where \(-\pi \leq \Omega \leq \pi\) with \( \Omega = 2\pi k/N \) is the conventional uniform frequency scale. Particularly, in the case of \( \alpha \) is a real parameter, the mapping is given by:
\[ \Omega_w = \Omega + 2 \arctan \left( \frac{1 - \alpha}{1 + \alpha} \tan \left( \frac{\Omega}{2} \right) \right) \]  
\[ \text{(14)} \]

The magnitude of the parameter \( \alpha \) is selected according to the percentage of frequency points to be concentrated inside the spectral lobe related with the frequency of interest, according to the following empirical rule:
\[ |\alpha| = \beta + \gamma \log_{10} N \]  
\[ \text{(15)} \]

where the values of \( \beta \) and \( \gamma \) are shown in table 1. In figures 1.a y 1.b, a conventional uniform frequency scale and an example of a warped non-uniform frequency scale on the unitary circle are shown respectively.

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Table 1.- Values of \( \beta \) and \( \gamma \) according to the percentage of frequency points concentrated in a spectral lobe [4][5].
5. Warped Discrete Time Frequency Distributions

In this work, a warped frequency scale is incorporated to the discrete time frequency distributions with periodic extension. Although this procedure is illustrated for the Wigner Ville distribution, its generalisation can be directly generated. The procedure essentially consists on the following: to calculate the warped discrete Fourier transform of the periodic extension of the generalised local auto-correlation function, instead of calculating its conventional discrete Fourier transform. Then, the warped discrete Wigner Ville time frequency distribution (WTFD\text{WV}) is:

\[ \text{WTFD}_{\text{WV}}(0,k) = 2 \sum_{t=0}^{L-1} R_\alpha(t) \left[ e^{j \frac{-\alpha t}{2}} + e^{j \frac{\alpha t}{2}} \right] \]  

(16)

where \( k = 0, \ldots, \hat{L} - 1 \) and the signal \( \hat{R}_\alpha(t) \) is the periodic extension of the generalised local auto-correlation function (8).

6. Frequency Estimation using TFD with Periodic Extension

To estimate the instantaneous frequency of an analytic signal using the Wigner Ville time frequency distribution with periodic extension [6], the following expression is used:

\[ f_i = \frac{\sum_{k=0}^{M/2-1} k \cdot \text{PTFD}_{\text{WV}}(0,k)}{\sum_{k=0}^{M/2-1} \text{PTFD}_{\text{WV}}(0,k)} \]  

(17)

which calculates the centroid respect to the frequency for each trace of constant time of the time frequency distribution.

7. Frequency Estimation using the Warped TFD

The procedure to estimate the instantaneous frequency of a signal with a dominating single frequency and a narrow bandwidth follows [5]. Notice that expression (17) cannot be used because the warped discrete Fourier transform does not satisfy the Parceval equality.

Initially, the conventional discrete Fourier transform of the signal is calculated:

\[ \text{DFT}(k) = \sum_{n=0}^{N-1} x(n) e^{j \frac{2\pi kn}{N}} \]  

(18)

Then, the value of the parameter \( \alpha = |\alpha| \exp(j\varphi) \) is calculated. For this, the phase associated to the frequency component with maximum magnitude is obtained. This locates coarsely the spectral lobe of interest. The obtained phase is the angle \( \varphi \) of the parameter \( \alpha \), defined by:

\[ \varphi = \text{angle} \left[ \max \left| \text{DFT}(k) \right|_{k=0}^{N-1} \right] \]  

(19)

The percentage of frequency points to be concentrated in that lobe is chosen (Table 1). Then, the magnitude of the parameter \( \alpha \) is calculated according to (15). The percentage can be chosen depending on the desired approach, the larger the percentage, the better the approach.

The warped discrete time frequency distribution of the signal is then calculated according to (16), using \( |\alpha| \).

Finally, a parabola is adjusted that includes the point associated with the frequency component maximum magnitude and its neighbouring points. Then, the instantaneous frequency is the vertex of that parabola. In figures 2.a and 2.b, this procedure is illustrated using a simple example.

8. Application to Doppler Flow Measurement

It is known that the mean velocity of the blood flow through the cross section of a vessel is proportional to the instantaneous frequency of a Doppler ultrasonic signal. In this work, a carotid artery signal is considered for this study. The simulation of that signal is detailed in [6][7][8]. The theoretical instantaneous frequency is shown in figure 3. The sampling frequency used is 25.5KHz. Sampling windows with maximum overlapping and length equal to 127 are used. For each one of those sampling windows, the instantaneous frequency is calculated using both, the discrete Wigner Ville time frequency distribution with periodic extension, PTFD\text{WV} (11), and the warped discrete Wigner Ville time frequency distribution, WTFD\text{WV} (16). The procedure to calculate the instantaneous frequency has been described in the sections 6 and 7 respectively. Finally, the RMS error respect to the theoretical instantaneous frequency is calculated.

9. Results

Figure 4 depicts a graph with the estimated instantaneous frequency using the Wigner Ville warped discrete time frequency distribution. In figure 5, the spectrum of a current window is shown (energy vs. frequency, with constant time). The results are compared for both distributions (distribution with periodic extension and using the warped distribution). It can be observed the concentration of the frequency points around the instantaneous frequency for the warped distribution. Table 2 shows the RMS errors in the estimation of the instantaneous frequency for the simulated signal. Different levels of normalised gaussian noise (SNR of 40, 30, 20 and 10 dB) have been added, with concentrations...
of frequency points around the instantaneous frequency of 30, 40, 50, 60 and 70 %. Figure 6 depicts a graph for the RMS instantaneous frequency estimation error for both approaches. The precision of the warped distribution to estimate the instantaneous frequency is notoriously better in the presence of noise, particularly above 30 dB.

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Table 2.- RMS error [Hz] obtained in the estimation of the instantaneous frequency of the simulated signal.

10. Conclusion

This approach presented in this paper incorporates a warped frequency scale (non-uniform) to the Cohen class time frequency distributions. Particularly, the warped discrete Wigner Ville time frequency distribution has been developed, although this procedure can be extended easily to other distributions. The method is applied to estimate the mean velocity of the blood flow through a vessel, which is proportional to the instantaneous frequency of the ultrasound signal obtained in the process of Doppler flow measurement. The experiments have been carried out using a Carotid artery simulated signal. The results obtained by the warped discrete time frequency distribution are compared with those obtained by the discrete Wigner Ville time frequency distribution with periodic extension. Results show that the distribution with periodic extension is easier to calculate since FFT-like algorithms of complexity $O(N \log N)$ are used; while the warped distribution uses algorithms which are based on the matrix multiplication whose complexity are $O(N^2)$. Nevertheless, the precision of the warped distribution to estimate the instantaneous frequency is notoriously better in the presence of noise, particularly above a SNR of 30 dB. The RMS error in the estimation using the warped discrete time-frequency distribution remains constant because it behaves as a band-pass filter located on the spectral lobe around the instantaneous frequency, eliminating the noise.

Acknowledgements

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References


Figure 1 a. Conventional uniform frequency scale on the unitary circle.
Figure 1 b. Warped non-uniform frequency scale on the unitary circle.

Figure 2 a. Magnitude of the discrete Fourier transform of a simple example (sine function with 1.43 rad). Note the coarsely precision.

Figure 2 b. Magnitude of the warped discrete Wigner Ville time frequency distribution of a simple example (sine function with 1.43 rad) with the adjusted parabola. Note the finer precision.

Figure 3. Theoretical instantaneous frequency of the simulated signal.

Figure 4. Estimated instantaneous frequency of the simulated signal using the warped discrete Wigner Ville time frequency distribution.

Figure 5. The spectrum (energy vs. frequency, with constant time) of a current analysed window using a Wigner Ville distribution with periodical extension (up) and a warped Wigner Ville distribution (down). Note the agglomerated frequential points.
Figure 6. RMS instantaneous estimation error [Hz]. The precision of the warped distribution to estimate the instantaneous frequency is better in presence of noise.