MULTIVARIABLE FRACTIONAL ORDER PI CONTROLLER FOR TIME DELAY PROCESSES

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ABSTRACT
Fractional order calculus has been used intensively to control various types of processes, with a focus on single-input-single-output systems. A few design procedures for multivariable systems exist. This paper proposes a simple and effective method for designing a multivariable fractional order PI controller for systems with multiple time delays. The design procedure is based on a steady state decoupling of the multivariable system and on settling time and gain robustness specifications. To avoid the time consuming optimization routines for determining the controller parameters, an iterative procedure is preferred and used. The case study presented demonstrates the efficiency of the proposed control design, the closed loop system behaving robustly to significant gain variations ranging ±30%.

KEY WORDS
Multivariable fractional order PI, Smith Predictor, decoupling, robustness

1. Introduction

The design problem of fractional order controllers has been the interest of many authors, with some valuable works [1,2,3,4] in which the fractional order (FO) controllers have been applied to a variety of dynamical processes, including integer-order and FO systems, so as to enhance the robustness and performance of the control systems [2,5]. The choice of fractional order PIμDλ controllers is based on their potential to improve the control performance [6,7,8], due to the supplementary tuning variables involved, μ and λ.

However, the majority of the works conducted in the field of fractional order controllers deal with plants of the form of single input single output systems, with only a few results obtained for multivariable FO PID controllers design [9, 10]. Previous techniques for designing the multivariable FO PID controllers consist in an approach similar to the single-input-single-output case. In [10], the novel method proposed by the authors is based on an H∞ problem with a controller structure constraint, while the controller parameters are optimized to achieve both user-specified robust stability and performance, the controller obtained being tested for controlling systems with multiple delays.

The present paper proposes a different approach in the design of a multivariable fractional order PI controller for systems with multiple time delays. The proposed method is simpler and offers significant robustness against gain uncertainties. The main outline of the controller design consists in a steady state decoupling of the process, design of the single-input-single-output (SISO) fractional order PI controllers for the decoupled process and computation of the final multivariable FO PI controller.

The design of the SISO FO PI controllers is based on a robustness specification to gain uncertainties and a settling time performance requirement, imposed using the gain crossover frequency. To avoid time consuming optimization routines, the authors use an iterative procedure to determine the tuning parameters of the FO PI controller, the proportional and integral gain and the fractional order. Since the design focuses on multivariable time delay processes, the control structure used consists in a Smith Predictor, which facilitates the design of the multivariable FO PI controller.

The paper is structured into four parts. Immediately after the Introduction, the proposed control method is presented, including the decoupling procedure, the multivariable fractional order PI controller design in a Smith Predictor structure and the implementation procedure of the controller obtained. Section 3 contains a case study to exemplify the tuning procedure described in Section 2. The design is presented step by step, while the last part of the section presents the simulation results that demonstrate the efficiency of the proposed control method. The last part of the paper contains the concluding remarks.

2. Multivariable Fractional Order Controller Design

2.1 Steady State Decoupling of Multivariable Processes

Given a general square process with ‘m’ inputs and ‘m’ outputs, the transfer function matrix may be written as:
\[ G_p(s) = \begin{bmatrix} g_{11}e^{-\tau_{1s}} & \cdots & g_{1m}e^{-\tau_{ms}} \\ \vdots & \ddots & \vdots \\ g_{m1}e^{-\tau_{ms}} & \cdots & g_{mm}e^{-\tau_{ms}} \end{bmatrix} \]  

(1)

where \( g_{ij} \) represent first order transfer functions from the \( j^{th} \) input to the \( i^{th} \) output \([11]\). It is assumed that the model of the multivariable process is equal to \( G_p(s) \):

\[ G_m(s) = G_p(s) \]  

(2)

The steady state gain matrix of the model \( G_m(s) \) is given by:

\[ G_m(s=0) = \begin{bmatrix} g_{10} & \cdots & g_{1m0} \\ \vdots & \ddots & \vdots \\ g_{m0} & \cdots & g_{mm0} \end{bmatrix} \]  

(3)

To design the controller, the first step consists in a steady state decoupling of the multivariable process \([11,12]\):

\[ G_D(s) = G_m(s) \cdot G_m^\# = \begin{bmatrix} g_{d11} & \cdots & g_{d1m} \\ \vdots & \ddots & \vdots \\ g_{dm1} & \cdots & g_{dmn} \end{bmatrix} \]  

(4)

where \( G_m^\# \) is the inverse of the steady state gain matrix in (3).

The decoupled process transfer function matrix \( G_D(s) \) consists of elements \( g_{dij} \), which are weighted sums of the original transfer functions of \( G_m(s), g_{ij}e^{-\tau_{is}} \). The decoupled process, in steady state conditions, will exhibit a zero value for each non-diagonal term and a unitary value for all diagonal terms, due to the static decoupling in (4). As a consequence, the non-diagonal terms will not be considered in the design of the controller.

The diagonal terms in \( G_D(s) \) consist of weighted sums of the various transfer functions in \( G_m(s) \). To facilitate the design of the controller, these diagonal terms are further approximated with simpler transfer functions. The approximation may be performed using various techniques, such as graphical methods or genetic algorithms \([11,13]\).

Assuming that the diagonal terms in \( G_D(s) \) have been approximated with transfer functions denoted as:

\[ g_{dii}^*(s) \approx g_{dii}^*(s) \]  

(5)

the next step consists in the actual design of the multivariable fractional order controller.

### 2.2 Multivariable Fractional Order PI Controller Design in a Smith Predictor Structure

The control structure proposed in this paper consists of the Smith Predictor, with multivariable fractional order PI primary controller. The control structure used is given in Fig. 1, where \( G_m(s) \) is the is the process model without the time delays and \( G_f(s) \) are feedback filters added to improve robustness to time delay variations \([14]\).

The general transfer function for a fractional order PI controller is given by:

\[ H_{FO-PI}(s) = k_p \left( 1 + \frac{k_i}{s^\mu} \right) \]  

(6)

where \( \mu \in (0,1) \) is the fractional order. The design of the controller consists in determining the controller parameters, \( k_p \) and \( k_i \), as well as the fractional order \( \mu \). Since the control structure consists in a Smith Predictor, the tuning of the parameters is done based on the delay free part of the transfer function in (5), denoted as \( g_{dii}^*(s) \).

![Figure 1. Proposed control structure](image_url)

The tuning of the fractional order PI controller is done independently for each \( g_{dii}^*(s) \), by imposing a gain crossover frequency, to establish the settling time of the closed loop \([15]\), and by imposing robustness to gain changes. Based on the imposed gain crossover frequency, the following equation is obtained:

\[ H_d(j\omega_{gc}) = 1 \]  

(7)

where \( \omega_{gc} \) is the gain crossover frequency. Equation (7) may be rewritten as:

\[ H_{FO-PI}(j\omega_{gc}) \cdot g_{dii}^*(j\omega_{gc}) = 1 \]  

(8)

If \( L \) is the imaginary part of \( g_{dii}^*(j\omega_{gc}) \) and \( K \) is the real part of \( g_{dii}^*(j\omega_{gc}) \), then equation (8) yields:

\[ \frac{1}{K + jL} \left( k_p \left[ 1 + k_i \omega_{gc}^{-\mu} \left( \cos \frac{\mu\pi}{2} - j\sin \frac{\mu\pi}{2} \right) \right] \right) = 1 \]  

(9)
leading to:

$$k_p = \frac{L^2 + K^2}{\sqrt{1 + 2k_i\omega_{gc}^2 \cos \frac{\pi \mu}{2} + k_i^2\omega_{gc}^{-2\mu}}}$$  \hspace{1cm} (10)$$

The gain robustness condition may be written as:

$$d(\angle H_d(j\omega_{gc})) \over d\omega_{gc} = 0$$  \hspace{1cm} (11)$$

The phase of $H_d(j\omega_{gc})$ is given by:

$$\angle H_d(j\omega_{gc}) = \tan^{-1}\left(-\frac{k_i\omega_{gc}^{-\mu} \sin \frac{\pi \mu}{2}}{1 + k_i\omega_{gc}^{-\mu} \cos \frac{\pi \mu}{2}}\right) - \tan^{-1}\left(\frac{L}{K}\right)$$  \hspace{1cm} (12)$$

Taking the derivative of (12) with respect to $\omega_{gc}$, yields:

$$\frac{\mu k_i \omega_{gc}^{-\mu-1} \sin \frac{\pi \mu}{2}}{1 + 2k_i\omega_{gc}^{-\mu} \cos \frac{\pi \mu}{2} + k_i^2\omega_{gc}^{-2\mu}} - \frac{L K - L \dot{K}}{L^2 + K^2} = 0$$  \hspace{1cm} (13)$$

Using (10) and (13), $k_p$ and $k_i$ can be uniquely determined, for a given fractional order $\mu$.

The tuning algorithm may be described as:

for $\mu = 0:1$

compute $k_i$ using (13)
compute $k_p$ using (10)
plot($\mu$, $[k_i, k_p]$)
select $\mu_b$ such that the curves for $k_i$ and $k_p$ intersect
compute final values for $k_p$ and $k_i$ using $\mu_b$
end

The continuous form of the final controller in Fig. 1 is then obtained using [13]:

$$G_C(s) = G_m\begin{bmatrix} H_{FO-PI_1}(s) & 0 & \ldots & 0 \\ 0 & H_{FO-PI_1}(s) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & H_{FO-PI_m}(s) \end{bmatrix}$$  \hspace{1cm} (14)$$

2.3 Implementation of Fractional Order PI Controller

The fractional order PI in (6), with $\mu \in (0,1)$, is implemented as:

$$H_{FO-PI}(s) = k_p \left(1 + k_i s^{1-\mu}\right)$$  \hspace{1cm} (15)$$

to ensure the effect of an integer order integrator, both at high and low frequencies [16].

Next, the fractional order PI controllers given by (15) are discretized [17] with the sampling period $T$:

$$s^\mu = \left(\frac{2}{T}\right)^\mu \frac{A(z^{-1}, \mu)}{A(z^{-1}, -\mu)}$$  \hspace{1cm} (16)$$

with

$$A(z^{-1}, \mu) = -\frac{\mu}{5}z^{-5} + \frac{\mu^2}{5}z^{-4} - \left(\frac{\mu}{3} + \frac{\mu^3}{15}\right)z^{-3} + \frac{2}{5}\mu^2z^{-2} - \mu z^{-1} + 1$$  \hspace{1cm} (17)$$

Using the discretization method in (16), the $G_C(z)$ controller in Fig. 1 is obtained.

3. Case Study: $^{13}$C Isotope Separation Column

To exemplify the algorithm described in Section 2, a case study consisting of a multivariable time delay process is considered [12]:

$$\begin{align*}
    y_1 &= g_{11}(s)e^{-\tau_{11}s} + g_{12}(s)e^{-\tau_{12}s} + 0u_1 \\
    y_2 &= g_{21}(s)e^{-\tau_{21}s} + g_{22}(s)e^{-\tau_{22}s} + g_{23}(s)u_2 \\
    y_3 &= g_{31}(s)e^{-\tau_{31}s} + g_{32}(s)e^{-\tau_{32}s} + g_{33}(s)u_3
\end{align*}$$  \hspace{1cm} (18)$$

with $g_{11}(s) = -0.1111 + s + 1.094s + 0.08423$, $g_{23}(s) = -1.104 + s + 0.1176$

$$g_{12}(s) = \frac{0.1152}{s^2 + 1.211s + 0.2021}, \quad g_{33}(s) = \frac{8.457}{s + 0.9851} - 0.001731$$

$$g_{21}(s) = \frac{0.003846}{s^2 + 0.1343s + 0.001961}, \quad g_{31}(s) = -0.009918$$

$$g_{22}(s) = \frac{0.05452}{s^2 + 1.056s + 0.0936}, \quad g_{32}(s) = \frac{0.006288}{s^2 + 1.085s + 0.09851}.$$

The steady state gain matrix is [12]:

$$G_m(s = 0) = \begin{bmatrix} -1.318 & 0.569 & 0 \\ -0.882 & -0.882 & -9.386 \\ -0.140 & 0.063 & 8.585 \end{bmatrix}$$  \hspace{1cm} (19)$$

with its inverse being equal to:
The time delays matrix of the MIMO system in (18) is given in Table 1, with values given in minutes.

<table>
<thead>
<tr>
<th></th>
<th>( U_1 )</th>
<th>( U_2 )</th>
<th>( U_3 )</th>
</tr>
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<tbody>
<tr>
<td>( y_1 )</td>
<td>10</td>
<td>32</td>
<td>0</td>
</tr>
<tr>
<td>( y_2 )</td>
<td>10</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>( y_3 )</td>
<td>18</td>
<td>35</td>
<td>0</td>
</tr>
</tbody>
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Table 1. Time delay matrix of the secondary processes (min)

The decoupled process transfer function matrix can then be computed using (4). The decoupled process open loop step responses are given in Fig. 2, where the new control inputs \( \mathbf{u}^*_1 \), \( \mathbf{u}^*_2 \) and \( \mathbf{u}^*_3 \) are weighted inputs of the original control inputs \( \mathbf{u}_1 \), \( \mathbf{u}_2 \) and \( \mathbf{u}_3 \), with the weights given in \( G_m^p \) matrix. The diagonal terms in the decoupled process transfer function matrix are given by:

\[
G_{d11} = -1.432 \cdot g_{11}(s)e^{-\tau_{11}s} - 1.558 \cdot g_{12}(s)e^{-\tau_{12}s}
\]

\[
G_{d22} = 0.856 \cdot g_{21}(s)e^{-\tau_{21}s} + 1.982 \cdot g_{22}(s)e^{-\tau_{22}s} + 6.7 \cdot 10^{-4} \cdot g_{23}(s)
\]

\[
G_{d33} = 0.936 \cdot g_{31}(s)e^{-\tau_{31}s} + 2.167 \cdot g_{32}(s)e^{-\tau_{32}s} + 0.115 \cdot g_{33}(s)
\]

The diagonal terms in (21) are approximated with the following transfer functions [12]:

\[
G_{d11}^*(s) = \frac{0.022274 \cdot e^{-6s}}{s^2 + 0.1044s + 0.02223}
\]

\[
G_{d22}^*(s) = \frac{0.0045918 \cdot e^{-8s}}{s^2 + 0.12s + 0.004592}
\]

\[
G_{d33}^*(s) = \frac{1.0565}{s + 1.057}
\]

Fig. 3 shows the step responses of transfer functions \( g_{d11}(s) \), \( g_{d22}(s) \) and \( g_{d33}(s) \), as well as those of the transfer functions in (22). The results show that the approximations in (22) have a similar step response with the original transfer functions of the decoupled process, given in (21).

The next step consists in the actual design of the fractional order PI controllers. For each transfer function in (22), a fractional order controller is designed by imposing a specific gain crossover frequency:

\[
\omega_{gc1} = 0.01
\]

\[
\omega_{gc2} = 0.001
\]

\[
\omega_{gc3} = 0.5
\]

Using the algorithm described in Section 2, Fig. 4, 5 and 6 are obtained, showing the values of the \( k_p \) and \( k_i \) parameters as a function of the fractional order \( \mu \).
Based on the plots in Fig. 4, 5 and 6, the fractional order is obtained as:

\[ \mu_1 = 0.364, \mu_2 = 0.276 \text{ and } \mu_3 = 0.925 \]  \hspace{1cm} (24)

leading to:

\[ k_{p1} = 0.35 \text{ and } k_{i1} = 0.363 \]
\[ k_{p2} = 0.323 \text{ and } k_{i2} = 0.32 \]
\[ k_{p3} = 0.596 \text{ and } k_{i3} = 0.765 \]  \hspace{1cm} (25)

By replacing the results in (25) and (24) in (15), the three individual fractional order PI controllers are obtained as:

\[
H_{\text{FO-PI}_1}(s) = 0.35 \left( 1 + \frac{0.363s^{1-0.364}}{s} \right) \\
H_{\text{FO-PI}_2}(s) = 0.323 \left( 1 + \frac{0.328s^{1-0.276}}{s} \right) \\
H_{\text{FO-PI}_3}(s) = 0.596 \left( 1 + \frac{0.765s^{1-0.925}}{s} \right) \]  \hspace{1cm} (26)

The continuous form of the final controller in (14) is thus computed as:

\[
G_C(s) = G_m^\# \begin{pmatrix} H_{\text{FO-PI}_1}(s) & 0 & 0 \\ 0 & H_{\text{FO-PI}_2}(s) & 0 \\ 0 & 0 & H_{\text{FO-PI}_3}(s) \end{pmatrix} \]  \hspace{1cm} (27)

with the matrix \( G_m^\# \) given in (20).

Using (16) and (17), the discrete forms of the fractional order PI controllers in (26) are obtained:

\[
s^{1-0.364} = \left( \frac{2}{T} \right)^{1-0.364} \frac{A(z^{-1},1-0.364)}{A(z^{-1},1-0.364)} \\
s^{1-0.276} = \left( \frac{2}{T} \right)^{1-0.276} \frac{A(z^{-1},1-0.276)}{A(z^{-1},1-0.276)} \\
s^{1-0.925} = \left( \frac{2}{T} \right)^{1-0.925} \frac{A(z^{-1},1-0.925)}{A(z^{-1},1-0.925)} \]  \hspace{1cm} (28)

with a sampling time \( T=0.3 \) minutes and:

\[
A(z^{-1},1-0.364) = -0.1273z^{-5} + 0.081z^{-4} - 0.2993z^{-3} + 0.162z^{-2} - 0.636z^{-1} + 1 \]
\[
A(z^{-1},1-0.276) = -0.1448z^{-5} + 0.1048z^{-4} - 0.2666z^{-3} + 0.2097z^{-2} - 0.724z^{-1} + 1 \]
\[
A(z^{-1},1-0.925) = -0.0150z^{-5} + 0.0011z^{-4} - 0.025z^{-3} + 0.0022z^{-2} - 0.075z^{-1} + 1 \]  \hspace{1cm} (29)

Fig. 7, 8 and 9 present the three outputs evolution, considering a step change in the second output reference signal, while Fig. 10, 11 and 12 show the evolution of the three inputs.

The nominal case scenario, considering \( G_p(s)=G_m(s) \), as well as situations considering ±30% gain variations are presented in the figures. The results obtained clearly indicate that the robustness specification regarding gain uncertainties is attained, with the maximum overshoot being less than 10%.
Figure 7. Output $y_1$ evolution considering a step change in the $y_2$ reference

Figure 8. Output $y_2$ evolution considering a step change in its reference signal

Figure 9. Output $y_3$ evolution considering a step change in the $y_2$ reference

Figure 10. Input $u_1$ evolution considering a step change in the $y_2$ reference

Figure 11. Input $u_2$ evolution considering a step change in the $y_2$ reference

Figure 12. Input $u_3$ evolution considering a step change in the $y_2$ reference
4. Conclusion

The main purposes of the paper was to provide a simple and efficient method for designing multivariable fractional order PI controllers with increased robustness against gain uncertainties. The case study presented demonstrates that the proposed method can achieve good results both under nominal conditions, as well as in situations of process gain variations.

Further research will be conducted to include in the performance specifications set the conditions for robustness regarding time delay variations, as well as possibilities to reduce the various input-output interactions.

Acknowledgement

This work was supported by a grant of the Romanian National Authority for Scientific Research, CNDDI–UEFISCDI, project number PN-II-PT-PCCA-2011-3.2-0591 155/2012.

References