NOISELET ENCODED COMPRESSIVE SENSING PARALLEL MRI

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ABSTRACT
Compressed sensing (CS) reconstruction relies on the sparsity of the signal in the transform domain and on the incoherence between sensing and sparsifying transform matrices. In CS-MRI, the sensing matrix is the randomly undersampled Discrete Fourier transform (DFT) matrix while Wavelet is used as the sparsifying transform. However the incoherence between the DFT and the Wavelet transform matrices is suboptimal for CS-MRI. In this paper we investigated the use of Noiselets as sensing matrix in MRI in order to improve the incoherence between sensing and sparsifying transform matrices. Noiselet basis are totally incompressible by Wavelets and spread out energy of the Wavelets in the Noiselet domain. In this work the k-space is encoded with Noiselet basis in the primary phase encode direction and a few random phase encodes are taken for the CS reconstruction. We compared the CS reconstruction error with uniform undersampling of the Fourier encoded and the Noiselet encoded MR images for various reduction factors in simulation, and showed that Noiselet encoded MRI performs better than Fourier encoded MRI. However for pseudo random undersampling in the Fourier domain and uniform random undersampling in the Noiselet domain both techniques perform equally well. However when both Noiselet encoded and Fourier encoded CS-MRI techniques were combined with parallel imaging using distributed compressed sensing model, the Noiselet encoded CS-MRI with uniform random undersampling outperforms the Fourier encoded CS-MRI with pseudo random undersampling. A tailored spin echo sequence is proposed to encode primary phase encode direction with Noiselet basis for MR imaging.

KEY WORDS
Magnetic Resonance Imaging, Noiselets, Compressed Sensing, Parallel Imaging, Non Fourier Encoding

1 Introduction
For many decades the world of digital signal processing has been governed by the well-known Shannon-Nyquist sampling theorem, which states that a band-limited signal must be sampled at least twice the bandwidth of the signal of interest for perfect reconstruction. This theorem do not consider any prior information about the signal of interest and performs equally well for all band-limited signals. In 2004 [1] a new theory of compressed sensing was introduced which exploits the sparsity of the signal of interest as a prior information to reconstruct the signal of interest sampled well below the Nyquist rate. According to CS theory, instead of measuring the sparse signal of interest directly, if only a few random linear combination of the signal is taken then these measurement will have enough information to reconstruct the signal of interest with appropriate non linear reconstruction. In real practice, the signal of interest may not be sparse in the acquisition domain, but CS can still be applied if the signal is sparse in some known transform domain. Compressed sensing requires certain mathematical properties to be satisfied to guarantee perfect reconstruction from highly undersampled data [2-4]. The three major requirements for perfect CS reconstruction are (i) sparsity of signal in known transform domain (ii) high incoherence between sensing and sparsifying transform matrices and (iii) a good non linear reconstruction algorithm.

Magnetic Resonance Imaging is a naturally fit system [5-6] for compressed sensing as it already acquires data in encoded form. In MRI, the Fourier domain data acquired is called k-space and much of the initial CS work was motivated by the fact that an image can be perfectly reconstructed by acquiring highly undersampled data in the Fourier domain [2]. Various schemes have been proposed to exploit CS in MRI [7-12]. A practical random sampling strategy to implement CS in MRI was suggested in [6]. Parallel imaging techniques were also combined with Fourier undersampled CS MRI [13-16]. CS-MRI with random encoding were also proposed to emulate the universal encoding defined by compressed sensing literature [17-18]. The first requirement of CS is well satisfied by most MR images as they are sparse in the Wavelet domain and some images such as Angiography images are sparse in pixel domain itself. However the second requirement of CS is weakly satisfied in MRI as Fourier and Wavelet transform matrices are not optimally incoherent.

In this paper we study Noiselet basis that are shown to be maximally incoherent with the Haar Wavelet in [19-20]. Noiselet basis are well suited for CS application [19] as it totally spread out energy of the signal of interest in the measurement domain. In [3] it was shown that the min-
imum number of measurement required for perfect CS reconstruction depends on the square of incoherence between measurement matrix and sparsifying transform matrix. In [19-20] it was shown that Noiselet are perfectly incoherent with Haar Wavelet and are best suited for CS application. In order to exploit the better incoherence offered by the Noiselet, in this work we show the use of Noiselets as the sensing matrix for CS data acquisition and reconstruction. To assess the performance of the Noiselet encoded MRI compared to the Fourier encoded MRI, we compared the mean square error (MSE) and signal to noise (SNR) ratio of the CS reconstructed images in simulations. Both the encoding schemes were also combined with parallel imaging and compared for reconstruction error. The simulation results show improvement in MSE and SNR with Noiselet encoded MRI compared to Fourier encoded MRI. Difference images formed by subtracting reference image and CS reconstructed image shows that Noiselet encoded CS-MRI captures fine details of the image more efficiently than the Fourier encoded CS-MRI. For practical implementation of Noiselet encoding we propose a tailored spin echo sequence to encode primary phase encode (PE) direction with Noiselet basis for MR imaging.

2 Background

2.1 Compressed Sensing

Compressed Sensing is essentially a method of solving an under-determined system of linear equations using some prior information about signal of interest. Consider a signal of interest $x \in C^n$ which is $S$-sparse, meaning it has at most $S$ non-zero values. The acquisition system do not acquire the signal $x$ directly but instead it acquires a linear combination

\[ y = \Phi x \quad \text{where} \quad y \in C^m \]

\[ \text{and} \quad m \ll n \]

Here $\Phi$ is an $m \times n$ measurement matrix. In MRI it is usually the randomly undersampled Fourier transform matrix. The signal of interest $x$ can be reconstructed with the convex minimization

\[ \min_x \| \hat{x} \|_1 \quad \text{S.T.} \quad y = \Phi \hat{x} \]

In this kind of acquisition scheme the $n$ dimensional signal $x$ has been projected onto the $m$ dimensional space ($m \ll n$). If this transformation of the signal of interest $x$ onto the lower dimensional space follows a mathematical property called Restricted Isometry Property (RIP) [21], then it is possible to recover signal $x \in C^n$ from $y \in C^m$ using a nonlinear reconstruction. The RIP is mathematically defined as

\[ (1 - \delta_S)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_S)\|x\|_2^2 \]

where $\delta_S$ is a small constant called RIP constant and lies between $[0,1]$. The RIP is in essence the conservation of energy and enforce that the magnitude of the signal must not change more than $\pm \delta_S$ in the $m$ dimensional space. The RIP constant determines the extent to which signal magnitude can change, after transformation of signal from $n$ dimensional space to $m$ dimensional space. Various bounds on the RIP constant have been proposed in [21], but explicit calculation of RIP constant is NP hard problem.

The signal $x$ may not be sparse but can still be recovered if it is sparse in some known transform domain. Consider a signal $x$ that is not sparse but its transformation $s = \Psi x$ is sparse; where $x, s \in C^n$ and $\Psi$ is $n \times n$ sparsifying transform matrix. Then the recovery problem can be modified to minimizing the $l_1$ norm of the sparse representation $s$ of the signal $x$

\[ \min_x \|\Psi \hat{x}\|_1 \quad \text{S.T.} \quad y = \Phi \hat{x} \]

The data acquisition process can be modeled as

\[ y = \Phi \Psi^* s = \Theta s \]

Where $\Psi^*$ is the conjugate transpose of $\Psi$.

When the signal is not sparse itself but is sparse in transform domain, the incoherence between sensing and sparsifying transform matrices need to be considered. The incoherence is linked to RIP and is an indirect measurement of RIP. The incoherence parameter determines the minimum number of samples needed to reconstruct the signal from incomplete linear measurement. In [3] showed that given the $m$ randomly selected measurement from $\Theta s$, the $l_1$ minimization can recover sparse signal $s$ if the number of measurement $m$ exceeds

\[ m \geq \text{Const.} \mu^2.S.log(n) \]

Where $S$ is the number of non zeros coefficient in sparse representation of signal of interest, $\text{Const.}$ is a very small constant and $\mu$ is the coherence parameter, the smaller the value of $\mu$, the lesser the number of random measurement required. The minimal coherence is termed as maximal incoherence that implies minimum value of $\mu$.

2.2 Noiselets

Noiselets are functions which are noise-like in the sense that they are totally incompressible by orthogonal wavelet packet methods [19-20]. Noiselet basis are constructed in a similar way as the Wavelet bases. First let us see the construction of the Noiselet basis and then we will look at the analogy between Wavelet and Noiselet. The Wavelet basis are constructed with translates and dilates of the mother wavelet function and described mathematically as
\( W_0(x) = \chi(x) \)
\( W_{2n}(x) = W_n(2x) + W_n(2x-1) \)
\( W_{2n+1}(x) = W_n(2x) - W_n(2x-1) \)

where,
\[ \chi(x) = \begin{cases} 1 & x \in [0,1) \\ 0 & \text{otherwise} \end{cases} \]

and \( W_0, \ldots, W_{2^n-1} \) are the orthonormal basis for the space \( V_n \). Similarly Noiselets are defined mathematically as
\[ f_1(x) = \chi(x) \]
\[ f_{2n}(x) = (1-i)f_n(2x) + (1+i)f_n(2x-1) \]
\[ f_{2n+1}(x) = (1+i)f_n(2x) + (1-i)f_n(2x-1) \]

where \( f_{2^n}, \ldots, f_{2^{n+1}} \) are the orthonormal basis for the space \( V_n \). As wavelet are constructed by translates and dilates of mother wavelet function, Noiselets are constructed in the similar way but by twisting the translates and dilates [19].

### 2.3 Incoherence between Noiselet and Wavelet

The mutual incoherence between two orthogonal bases \( \Phi \) and \( \Psi \) is defined as
\[ \mu(\Phi, \Psi) = \max_{x} \| \langle \Phi, \Psi \rangle \|_2 \]

where \( \Phi \) and \( \Psi \) are orthogonal basis such that \( \Phi \times \Phi = nI \) and \( \Psi \times \Psi = I \). Consider a Noiselet basis \( \Phi \in C^{n \times n} \) such that \( \Phi \times \Phi = nI \) and a Harr wavelet basis \( \Psi \in R^{n \times n} \) such that \( \Psi \times \Psi = I \). Then the product \( U = \Phi \times \Psi \) will have entries of constant magnitude \( \| U \|_2 = 1 \), which implies that coherence \( \max_{x} \| \langle \Phi, \Psi \rangle \|_2 = 1 \). The value of coherence parameter lies between \( 1 \leq \mu < \sqrt{n} \). Here the value of coherence parameter between Noiselet and Wavelet is minimal and hence implies maximal incoherence between Noiselet and Wavelet. Apart from incoherence, Noiselet also poses two important properties.

- Noiselets are derived in the same way as wavelets, therefore it can be modeled as multi-scale filter-bank and can be applied in \( O(n.\log(n)) \).
- The real and imaginary part of Noiselet bases is binary valued. Therefore all values of Noiselet bases is from a set of four complex number that makes it easy for hardware implementation.

### 3 Proposed Noiselet Encoding Scheme

We propose using Noiselet bases as a measurement matrix for data acquisition in MRI to exploit better incoherence offered by Noiselet bases. The measurement process in Noiselet encoding is described by Eq.(5). Here \( \Phi \) is the measurement matrix which is formed by randomly selecting a few rows from the Noiselet matrix, \( \Psi \) is Daubechies 4 wavelet transformation matrix and \( x \) is the image to be acquired. We use modification of “SGPL1: A solver for large-scale sparse reconstruction” [22] to solve the following convex optimization problem
\[ \min_x \| \Psi.x \|_1 \quad \text{S.T.} \quad \| y - \Phi.x \|_2 \leq \epsilon \]

Where \( \epsilon \) is a small constant that determines the allowed noise level in the reconstructed image. In MRI, RF pulse sequence is used to acquire data. The pulse sequence uses RF pulse to excite certain volume of the human body and linear gradient waveform are used to encode the excited volume onto the Fourier domain. In [23-24], RF pulse sequence design is proposed to do non-Fourier encoding in MRI. We propose to tailor conventional spin echo sequence to acquiring the data in Noiselet domain. Fig.1 shows our proposed sequence diagram to acquire Noiselet encoded data. In this sequence we replace the slice selective Sinc RF pulse in conventional spin echo with Fourier transform of Noiselet bases along with constant linear gradient in phase encode (PE) direction. This will excite the whole volume and encode PE direction with Noiselet profile. After exciting Noiselet profile, a 180° refocusing RF pulse is used to select slice of interest followed by the readout gradient. For each TR, the excitation pulse is replaced by Fourier transform of new Noiselet basis function to acquire all required Noiselet coefficients in PE direction.

![Figure 1: Pulse sequence to acquire Noiselet encoded data.](image)

4 Simulations

Simulation are carried out to assess the performance of Noiselet encoding scheme and to compare with that of the Fourier encoding. Random undersampling is used in the simulations of the reference image. Comparison between Noiselet encoding and Fourier encoding with same number of measurements is done to examine the quality of reconstructions from Noiselet measurement and the Fourier measurement. Mean square error (MSE), signal to noise
ratio (SNR) and difference images are used to compare the quality of reconstructions. Random undersampling is performed only along phase encode direction because undersampling in only one direction is practically feasible in 2D MRI data acquisition. Simulations are carried out for the following three scenarios.

- Uniform random undersampling as per compressed sensing theory
- Pseudo random undersampling [6] to capture more energy by densely sampling the center of k-space and sparsely sampling the outer regions of k-space.
- Pseudo random undersampling combined with parallel imaging as suggested in [15-16] according to distributed compressed sensing frame work.

Figure 2: Sampling pattern for (a). Uniform undersampling and (b). Pseudo random undersampling

Fig.2(a) shows the the sampling pattern for uniform random undersampling where the k-space is sampled according to uniform probability distribution. Fig.2(b) shows the pseudo random undersampling pattern where the center of k-space is densely sampled and the sampling becomes sparse towards the outer k-space according to the Gaussian probability distribution function.

Mean square error is calculated as \( \text{MSE} = \frac{1}{n} \sum_{i=1}^{N} (\| x_i - m_i \|^2) \) where \( N \) = total number of pixel in the reconstructed image, \( x_i \) = reference image, \( m_i \) = reconstructed image. SNR is calculated as SNR = \( 10 \log (P_{\text{signal}}/P_{\text{noise}}) = 10 \log (\text{var}/\text{mse}) \), where, \( \text{var} = \frac{1}{N} \| m - \bar{m} \|^2, \bar{m} = \frac{1}{N} \sum_{i=1}^{N} \| x_i \|. \)

4.1 Uniform Random Undersampling

A fully sampled brain image (256x256) is used as the reference image for all the simulations. Fourier transform of this image is taken to get Fourier encoded k-space data followed by uniform undersampling (pattern shown in Fig.2(a)). SPGL solver is used to solve the convex optimization problem and image is reconstructed from the data undersampled by a factor of 2. For Noiselet encoding simulations we took Noiselet transform in one direction and

Figure 3: (a) Reference image. (b) Reconstructed image with Fourier encoding. (c) Difference image for Fourier encoding. (d) Reconstructed image with Noiselet encoding. (e) Difference image for Fourier encoding.

Table 1: MSE and SNR for uniform random undersampling.

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<thead>
<tr>
<th>Reduction Factor</th>
<th>Fourier Encoding</th>
<th>Noiselet Encoding</th>
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<tbody>
<tr>
<td>2</td>
<td>MSE = 0.0073</td>
<td>MSE = 0.0012</td>
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<tr>
<td></td>
<td>SNR = 16.8271</td>
<td>SNR = 23.4706</td>
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</table>

4.2 Pseudo Random Undersampling

Pseudo random undersampling in CS-MRI is know to perform better than uniform random undersampling [6]. Fig.2(b) shows the sampling pattern of this sampling scheme which samples the center of k-space densely compared to the outer part of k-space. This acquisition is motivated by the fact that in k-space most of the energy is concentrated in the center. By densely sampling the center,
most of the energy is captured and randomness is achieved by randomly sampling the outer region. In case of Noiselet encoding, the energy is totally spread out in Noiselet domain, therefore it is better to use uniform random undersampling. With pseudo random undersampling, Fourier encoding exploits an extra information about the energy distribution in the k-space that makes Fourier encoding perform better than uniform random undersampling.

Fig.4 compares the reconstructed and difference images for Fourier encoding with Pseudo random undersampling and Noiselet encoding with uniform undersampling. Fig.4(b) and (c) show the reconstructed images for the reduction factors of 2 and 3 respectively for Fourier encoding, Fig.4(f) and (g) are corresponding difference images. Fig.4(d) and (e) show the reconstructed images for reduction factors of 2 and 3 respectively for Noiselet encoding, Fig.4(h) and (i) are corresponding difference images. In this case the advantage offered by the better incoherence of Noiselets are undermined and both the encoding schemes perform more or less equally well. Table 2 shows the MSE and SNR for the acceleration factors of 2 and 3 for both the encoding scheme. It is evident here that Fourier encoding also performs as good as Noiselet encoding.

<table>
<thead>
<tr>
<th>Reduction Factor</th>
<th>Fourier Encoding</th>
<th>Noiselet Encoding</th>
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<tbody>
<tr>
<td>2</td>
<td>MSE = 0.0013</td>
<td>MSE = 0.0012</td>
</tr>
<tr>
<td></td>
<td>SNR = 22.8271</td>
<td>SNR = 23.4706</td>
</tr>
<tr>
<td>3</td>
<td>MSE = 0.0037</td>
<td>MSE = 0.0035</td>
</tr>
<tr>
<td></td>
<td>SNR = 18.8271</td>
<td>SNR = 19.9377</td>
</tr>
</tbody>
</table>

4.3 Pseudo Random Undersampling with Parallel Imaging

Compressed sensing has been combined with parallel imaging in a Distributed Compressed Sensing framework in [15]. Here we combine both Noiselet encoded CS-MRI and Fourier encoded CS-MRI with parallel imaging and compare their performance. The measurement process in MRI taking in account the coil sensitivity profile can be modeled as:

\[ y_i = \Phi \rho_i x \]

where \( i = \{1, 2, \ldots, l\} \), \( l \) is the number of receiver coil, \( \rho_i \) is the sensitivity profile for the \( i^\text{th} \) receive coils, \( y_i \) is the acquired data from the \( i^\text{th} \) receive coil and \( s = \Psi x \).

Let

\[ Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_l \end{bmatrix} \quad \text{and} \quad E = \Phi \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_l \end{bmatrix} \]

then Eq.11 can be written as \( Y = ES^\ast s \). In this case following minimization problem is solved to find the desired image \( x \).

\[ \min \| x \|_1 \quad \text{S.T.} \quad \| y - E x \|_2 \leq \epsilon \] (12)

Fig.5 shows the reconstructed and difference images (scaled by a factor of 4) for the simulations using Noiselet encoding and Fourier encoding schemes with parallel imaging. The sensitivity profiles of 8 coils are used in simulation. For Fourier encoding, Fig.5 (b), (c) and (d) show the reconstructed images for the reduction factors of 2, 3 and 4 respectively, and Fig.5 (e), (f) and (g) show the corresponding difference images. For Noiselet encoding, Fig.5 (h), (i) and (j) show the reconstructed images for the reduction factors of 2, 3 and 4 respectively, and Fig.5 (k), (l) and (m) show the corresponding difference images. It is evident from the difference images that the Noiselet encoding outperforms Fourier encoding. It is observed from the difference images that the Noiselet encoding is better in preserving the finer details in the image while Fourier encoding does not preserve the details. Table 3 shows the MSE and SNR for the acceleration factors of 2, 3 and 4 for both the encoding schemes. SNR of Noiselet encoding is approximately 4dB higher than Fourier encoding for reduction factor of 3 and 4. MSE of Noiselet encoding is approximately half than the Fourier encoding for reduction factor of 3 and 4. Moreover for the reduction factor of 2, SNR is 12dB higher in Noiselet encoding compared to Fourier encoding.
Table 3: MSE and SNR for Pseudo random undersampling with Parallel Imaging

<table>
<thead>
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<th>Reduction Factor</th>
<th>Fourier Encoding</th>
<th>Noiselet Encoding</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>MSE=$1.1547 \times 10^{-4}$&lt;br&gt;SNR=35.5025</td>
<td>MSE=$5.128 \times 10^{-6}$&lt;br&gt;SNR=47.4182</td>
</tr>
<tr>
<td>3</td>
<td>MSE=$3.3355 \times 10^{-4}$&lt;br&gt;SNR=30.7689</td>
<td>MSE=$1.2248 \times 10^{-4}$&lt;br&gt;SNR=34.8991</td>
</tr>
<tr>
<td>4</td>
<td>MSE=$8.6665 \times 10^{-4}$&lt;br&gt;SNR=24.1112</td>
<td>MSE=$4.1637 \times 10^{-4}$&lt;br&gt;SNR=28.5344</td>
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5 Discussion

It is evident from the simulation with uniform random undersampling that the maximal incoherence offered by the Noiselet results in better reconstruction for the Noiselet encoded MRI than the Fourier encoded MRI, as suggested theoretically in [3]. However in case of pseudo random undersampling, Fourier encoded MRI efficiently utilizes another information about the distribution of energy in k-space and tries to capture more energy by densely sampling the center of k-space. In this way Fourier encoded CS MRI sacrifices some of the randomness and captures more energy or a low resolution image better. In effect, even after sacrificing some of the randomness while capturing most of the energy, the reconstruction quality become better compared to uniform random undersampling. This extra information about the energy distribution utilized efficiently by Fourier encoded CS MRI makes it perform approximately as good as Noiselet encoded CS MRI. However when both the techniques are combined with parallel imaging in distributed compressed sensing framework, it is observed that Noiselet encoded MRI still outperforms Fourier encoded MRI. It is shown in [16] that when CS is combined with parallel imaging in distributed CS framework, the incoherence increases due to the sensitivity profiles of receive coils. Here we can logically infer that the increase in incoherence for the Noiselet encoded CS MRI with parallel imaging is more than that of Fourier encoded CS MRI with parallel imaging, and this makes Noiselet encoded CS-MRI to perform better than Fourier encoded CS-MRI.

It can be seen from the difference images in Fig.5 that Noiselets are better in capturing the fine details of image. This is due to the fact that Noiselets totally spreads out the energy of image, as shown in Fig.6(b) while in Fourier encoding energy is concentrated at the center as shown in Fig.6(a). Therefore each Noiselet coefficient contains some information about the coarse as well as fine features of image. While in Fourier encoded CS MRI, we densely sample the center of the k-space which contain only the coarse information, and this makes Fourier encoded CS-MRI less effective in capturing the finer details of desired image.

In [17] random RF pulses were used to emulate the universal encoding with the measurement matrix being derived from the uniform i.i.d Gaussian distribution. They stated that because the random measurement matrix is not unitary, perfect reconstruction is not possible even after acquiring all the coefficient, and it will result in noise amplification. Indeed Noiselet basis forms unitary matrix, therefore perfect reconstruction is possible in using Noiselet basis and it will not result in noise amplification. Therefore Noiselets are better and more powerful bases function for CS-MRI than simply using the random measurement matrix.

6 Conclusion

We have shown that Noiselet encoded MRI performs as well as the conventional Fourier encoded MRI. However when Noiselet encoded MRI is combined with parallel imaging in the distributed CS framework, it outperforms Fourier encoded MRI. It is found that Noiselet encoded
CS-MRI is efficient at capturing details of the image. Thus, a potential candidate for the applications requiring high resolution images.

References


