DESIGN OF QUASI-EQUIRIPPLE IIR FILTERS WITH PRESCRIBED FLATNESS AND APPROXIMATELY LINEAR PHASE

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ABSTRACT
In this paper, we propose a method for designing quasi-equiripple infinite impulse response (IIR) filters with prescribed flatness and approximately linear phase characteristics. In the proposed method, the flatness condition in stopband is preincorporated into the transfer function and the design problems are formulated as an iterative quadratic programming (QP) problem. The flatness condition in passband is added to the QP problem as the linear matrix equality. As a result, the proposed method can design the IIR filters which have an equiripple response and prescribed flatness in passband or stopband or both. Finally, we show some design examples and confirm the effectiveness of the proposed method.

KEY WORDS
IIR filter, Flat magnitude, Flat group delay, Equiripple characteristic, Quadratic programming.

1 Introduction
Infinite impulse response (IIR) digital filters are widely used in various fields such as communication, measurement, and image processing, and many design methods have been proposed [1]-[9]. Chebyshev type and inverse Chebyshev type filters [2]-[5] have an equiripple characteristic in either the passband or stopband, and a flat characteristic in the other band. These filters are effective for suppressing the ringing and the checkerboard distortion. In [2] and [3], the design methods based on the Remez algorithm have been proposed for the Chebyshev type and inverse Chebyshev type IIR filters with approximately linear phase characteristics. However, the filters that can be designed by these methods are limited because of the condition on setting the initial value. By using the linear semi-infinite programming and the extended positive realness, the design method of stable inverse Chebyshev type IIR filters with approximately linear phase characteristic have been proposed [4]. However, the problem size depends on the number of the discrete frequency points in this method.

In this paper, a new design method is proposed for quasi-equiripple filters with prescribed flatness and approximately linear phase characteristics. Magnitude flatness and multiple zeros are desirable in designing sample rate converters in order to suppress the alias components and the design of wavelet basis [10]. In the proposed method, the flat stopband characteristics are realized first by placing multiple zeros in the stopband. Then, the frequency characteristics are approximated under the least square criterion, by using the transfer function with the stopband flatness. The design problems are formulated as an iterative quadratic programming (QP) problem. At that time, the flatness conditions in the amplitude and group delay in passband is added to the constraints of the design problems using the linear matrix equality. The proposed method can design not only the Chebyshev and inverse Chebyshev filters but also simultaneous Chebyshev filters with the prescribed flatness in passband and stopband. The effectiveness of the proposed method is verified through some design examples.

2 IIR digital filters
The frequency response $H(e^{j\omega})$ of an IIR digital filter is defined as

$$H(e^{j\omega}) = \frac{A(e^{j\omega})}{B(e^{j\omega})} = \sum_{n=0}^{N} a_n e^{-jn\omega}$$

$$\sum_{m=0}^{M} b_m e^{-jm\omega}$$

where $N$ and $M$ are the orders of the numerator and denominator, respectively. The filter coefficients $a_n$ and $b_m$ are real, and $b_0 = 1$ in general. The desired frequency response $H_d(e^{j\omega})$ of a lowpass filter can be expressed as

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\tau_d\omega} & (0 \leq \omega \leq \omega_p) \\ 0 & (\omega_s \leq \omega \leq \pi) \end{cases}$$

where $\tau_d$ is a desired group delay in the passband and $\omega_p$ and $\omega_s$ are, respectively, passband and stopband edge angular frequencies. Then, the flatness conditions of the magnitude and group delay in the passband are given as follows [3]:

$$\frac{\partial^i |H(e^{j\omega})|}{\partial \omega^i} \bigg|_{\omega=0} = \begin{cases} 0 & (i = 0) \\ 1 & (i = 1, 2, \ldots, K_p - 1) \end{cases}$$

$$\frac{\partial^i \tau(\omega)}{\partial \omega^i} \bigg|_{\omega=0} = \begin{cases} \tau_d & (i = 0) \\ 0 & (i = 1, 2, \ldots, K_p - 2) \end{cases}$$
where $K_p$ is a parameter expressing the flatness in the passband. The magnitude flatness condition in the stopband is

$$\frac{\partial^i |\tilde{H}(e^{j\omega})|}{\partial \omega^i} \bigg|_{\omega=\pi} = 0 \quad (i = 0, 1, \ldots, K_s - 1) \quad (5)$$

where $K_s$ is a parameter expressing the flatness in the stopband.

### 3 Design of IIR lowpass filters

#### 3.1 Flatness condition in stopband

First, we consider the flatness condition in the stopband. Let $\tilde{H}(e^{j\omega})$ be a noncausal shifted version of $H(e^{j\omega})$;

$$\tilde{H}(e^{j\omega}) = H(e^{j\omega})e^{j\tau_d\omega} = \sum_{n=0}^{N} a_ne^{-j(n-\tau_d)\omega} + \sum_{m=0}^{M} b_me^{-jm\omega} \quad (6)$$

Then, the flatness condition in eq. (5) becomes

$$\frac{\partial^i |\tilde{H}(e^{j\omega})|}{\partial \omega^i} \bigg|_{\omega=\pi} = 0 \quad (i = 0, 1, \ldots, K_s - 1). \quad (7)$$

In order to meet the flatness condition of eq. (7), it is necessary to place $K_s$ multiple zeros at $\omega = \pi$. Hence, the frequency response $\tilde{H}(e^{j\omega})$ can be expressed as follows [2].

$$\tilde{H}(e^{j\omega}) = e^{j\omega}\frac{A(e^{j\omega})}{B(e^{j\omega})} = \left(1 + e^{-j\omega}\right)K_s \sum_{n=0}^{N-K_s} c_ne^{-j(n-\tau_d)\omega} + \sum_{m=0}^{M} b_me^{-jm\omega} \frac{\left(1 - e^{-j\omega}\right)^{K_p}}{1 + e^{-j\omega}} \quad (8)$$

**3.2 Flatness condition in passband**

With eq. (8) used, the flatness conditions in eqs. (3) and (4) become

$$\frac{\partial^i \tilde{H}(e^{j\omega})}{\partial \omega^i} \bigg|_{\omega=0} = \begin{cases} \frac{\partial^i A(e^{j\omega})}{\partial \omega^i} \bigg|_{\omega=0} & (i = 0) \\ 0 & (i = 1, 2, \ldots, K_p - 1) \end{cases} \quad (9)$$

Eq. (9) is the same as follows.

$$\frac{\partial^i \tilde{A}(e^{j\omega})}{\partial \omega^i} \bigg|_{\omega=0} = \frac{\partial^i B(e^{j\omega})}{\partial \omega^i} \bigg|_{\omega=0} \quad (i = 0, 1, \cdots, K_p - 1). \quad (10)$$

Consequently, we can get the following linear equation in matrix form:

$$Uh = V \quad (11)$$

where

$$h = [c_0, \ldots, c_{N-K_s}, b_1, \ldots, b_M]^T, \quad (12)$$

$$V = [1, 0, \ldots, 0]^T, \quad (13)$$

$$U = \begin{bmatrix} u(0, 0) & \ldots & u(0, N-K_s) \\ \vdots & \ddots & \vdots \\ u(K_p-1, 0) & \ldots & u(K_p, N-K_s) \end{bmatrix}, \quad (14)$$

$$U_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \ldots & 0 & \ldots & 0 \\ -1 & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ -1 & 0 & \ldots & 0 & \ldots & 0 \end{bmatrix}, \quad (15)$$

$$U_2 = \begin{bmatrix} -1^0 & \ldots & -M^0 \\ \vdots & \ddots & \vdots \\ -1^0 & \ldots & -M^0 \end{bmatrix}, \quad (16)$$

$$u(p, q) = \sum_{k=0}^{K_s} \sum_{r=0}^{N-K_s} c_k \sum_{s=0}^{M} C_s r^{p-k} (q - \tau_d)^k, \quad (17)$$

and $(\cdot)^T$ denotes the transpose of $(\cdot)$.

### 3.3 Formulating the design as a QP problem

If the frequency response of eq. (8) is used, it is found that the flatness characteristic at $\omega = \pi$ can easily be realized.

Here, let $\hat{H}_d(e^{j\omega})$ be the desired magnitude response of a lowpass filter, i.e.,

$$\hat{H}_d(e^{j\omega}) = \begin{cases} 1 & (0 \leq \omega \leq \omega_p) \\ 0 & (\omega_s \leq \omega \leq \pi) \end{cases}. \quad (18)$$

Using eqs. (8) and (18), the weighted least squares design problem is

$$\min_{c, b} J = \sum_{i=1}^{L} W(\omega_i)|\tilde{H}(e^{j\omega_i}) - \hat{H}_d(e^{j\omega_i})|^2 \quad (19)$$

where $L$ is the total number of grid points in the passband and stopband, $W(\omega_i)$ is the weighting function, and $\omega_i (l = 1, \ldots, L)$ are the discrete frequency points used in the calculation. However, it is difficult to solve eq. (19) directly because $\hat{H}(e^{j\omega_i})$ is a rational function. Thus, we use the following iterative design formula:

$$\min_{c, b} J = \sum_{i=1}^{L} W(\omega_i)|\hat{A}(e^{j\omega_i}) - \hat{H}_d(e^{j\omega_i})B(e^{j\omega_i})|B_{k-1}(e^{j\omega_i})|^2 \quad (20)$$

where $k$ is the number of the iterations.

After some manipulation, eq. (20) can be formulated as the following QP problem:

$$\min_{h_k} h_k^T \left( \text{Re}(P^T)W\text{Re}(P) + \text{Im}(P^T)W\text{Im}(P) \right) h_k - 2 \left( \text{Re}(Q^T)W\text{Re}(P) + \text{Im}(Q^T)W\text{Im}(P) \right) h_k \quad (21)$$
where
\[ P = \text{diag}(G) \begin{bmatrix} e^{-j\omega_1} & \ldots & e^{-j(N-K_s-\tau_d)\omega_1} \\ \vdots & \ddots & \vdots \\ e^{-j\omega_L} & \ldots & e^{-j(N-K_s-\tau_d)\omega_L} \end{bmatrix} , \] (22)
\[ Q = \text{diag}(d) \begin{bmatrix} e^{-j\omega_1} & \ldots & e^{-jM\omega_1} \\ \vdots & \ddots & \vdots \\ e^{-j\omega_L} & \ldots & e^{-jM\omega_L} \end{bmatrix} , \] (23)
\[ G = [(1+e^{-j(\omega_1)})K_s, \ldots ,(1+e^{-j(\omega_L)})K_s] , \] (24)
\[ d = [H_d(e^{j\omega_1}), \ldots ,H_d(e^{j\omega_L})] , \] (25)
\[ W = \text{diag} \left( \frac{W(\omega_1)}{|B_{k-1}(\omega_1)|^2}, \ldots , \frac{W(\omega_L)}{|B_{k-1}(\omega_L)|^2} \right) . \] (26)

3.4 Update of the weighting function \( W(\omega) \)

It has been well known that the filters obtained under weighted least square criterion have a large magnitude ripple near the band edges. So in order to realize the equiripple characteristics in the passband or stopband or both, the weighting function used at every iteration is adjusted using the modified Lawson’s method [11] and the QP problem is solved to obtain the coefficients. In this paper, the weighting function \( W(\omega) \) in \( k \)th iteration step is updated as follows:

\[ W_{k+1}(\omega) = \frac{W_k(\omega)\beta_k(\omega)}{\frac{1}{L} \sum_{l=1}^{L} W_k(\omega_l)\beta_l(\omega_l)} \] (27)

where the envelope function \( \beta_k(\omega) \) is given as the function of straight line formed by joining together all the extremal points of the same frequency band of interest on the error function which is expressed as

\[ E_k(\omega) = \left| H(e^{j\omega}) - \tilde{H}_d(e^{j\omega}) \right| . \] (28)

Using the extremal points \( \omega_i \) of \( E_k(\omega) \), \( \beta_k(\omega) \) can be calculated by

\[ \beta_k(\omega) = \frac{\omega - \tilde{\omega}_i}{\tilde{\omega}_{i+1} - \tilde{\omega}_i} E_k(\tilde{\omega}_{i+1}) + \frac{\tilde{\omega}_{i+1} - \omega}{\tilde{\omega}_{i+1} - \tilde{\omega}_i} E_k(\tilde{\omega}_i) \]

for \( \tilde{\omega}_i < \omega < \tilde{\omega}_{i+1} \) (29)

where \( \tilde{\omega}_i \) denotes the \( i \)th extremal frequency of the the error function \( E_k(\omega) \).

Thus, the design problem in \( k \)th iteration step becomes a standard quadratic programming (QP) problem below:

\[
\min_{\mathbf{h}_k} h_k^T \left( \text{Re}(P^T)W_k\text{Re}(P) + \text{Im}(P^T)W_k\text{Im}(P) \right) h_k
- 2 \left( \text{Re}(Q^T)W_k\text{Re}(P) + \text{Im}(Q^T)W_k\text{Im}(P) \right) h_k
\]

subject to \( \mathbf{U} h_k = \mathbf{V} \). (30)

The design procedure of the proposed method is summarized as follows. [Design Procedure]

**Step 0:** Set the desired magnitude response \( \tilde{H}_d(\omega) \), group delay response \( \tau_d \) filter order \( N \) and \( M \), flatness \( K_p \) and \( K_s \), passband edge \( \omega_p \), stopband edge \( \omega_s \), weighting function \( W(\omega) \), number of grid points \( L \) and \( R \).

**Step 1:** Solve the QP problem in eq. (30) to obtain the filter coefficient \( h_k \).

**Step 2:** If \( \frac{\sum(|h_k - h_{k-1}|)}{\sum(|h_k|)} \leq \epsilon \), then stop; otherwise, go to next.

**Step 3:** Update the weighting function \( W(\omega) \) using eqs. (27) - (29) and go back to Step 1.

4 Design examples

In this section, the some design examples are given to demonstrate the effectiveness of the proposed method. In all the following examples, \( R = 1000 \), \( \epsilon = 10^{-7} \), and \( \epsilon_R = 10^{-5} \). Moreover, “quadprog” function in MATLAB was used to solve the QP problem in eq. (30).

4.1 Example 1

We first show the design examples of the inverse Chebyshev type IIR filters which have a flat characteristic in passband and an equiripple characteristic in stopband. The design specifications are as follows: \( N = 12, M = 5, \omega_s = 0.50\pi, \tau_d = \{10.2, 12.0, 13.8\}, K_p = 10, K_s = 0 \). The total grid point \( L \) is 500 which is the sum of 0 point in the passband and 500 points in the stopband. The magnitude response and group delay response of the obtained filter are shown in Figure 1. The numerical performance are shown in Table 1 with them obtained by the conventional method [3] based on Remez algorithm. From Figure 1, it is confirmed that the magnitude and group delay responses both have a flat characteristic at \( \omega = 0 \) and the magnitude response in stopband is equiripple. From Table 1, it is confirmed that the resulting filters by the proposed method have almost the same or better characteristics compared with them by the conventional method.

Next, we consider the following specifications: \( N = 14, M = 9, \omega_s = 0.40\pi, \tau_d = 11.0, K_p = 9 \). In [3], the condition that \( K_p \geq M + 1 \) must be met in order to set the initial value. Therefore, this filter cannot be designed using [3]. The magnitude response and the group delay response of the filter obtained by the proposed method are shown in Figure 2. It is confirmed from Figure 2 that both of the magnitude and group delay responses of the obtained filter have a flat characteristic at \( \omega = 0 \) and the magnitude response in stopband is equiripple. The minimum stopband...
Figure 1. Inverse Chebyshev type IIR filters with $N = 12$ and $M = 5$ in example 1.

Table 1. Comparison between Proposed method and Ref.[3]

<table>
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<tr>
<th></th>
<th>Minimum Stopband Attenuation [dB]</th>
<th>Maximum Pole Radius</th>
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<td>10.2</td>
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<tr>
<td>13.8</td>
<td>58.34</td>
<td>0.7399</td>
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Figure 2. Inverse Chebyshev type IIR filter with $N = 14$ and $M = 9$ in example 1.

4.2 Example 2

We show the design examples of the Chebyshev type IIR filters which have an equiripple characteristic in passband and a flat characteristic in stopband. The design specifications are as follows: $N = 15, M = 6, \omega_p = 0.30\pi, \tau_d = 12.0, K_p = 0, K_s = \{9, 10, 11\}$. The total grid point $L$ is 500 which is the sum of 500 points in the passband and 0 point in the stopband. The magnitude response and group delay response of the obtained filter are shown in Figures 3(a)-(c). The performance of each filter is shown in Table 2 with them obtained by the conventional method [2] based on Remez algorithm. In Table 2, $R_p$ is the maximum magnitude error in passband, $G_{err}$ is the maximum group delay error in passband, and $P_{\text{pole}}$ is the maximum pole radius. The complex magnitude error and pole-zero plot of the filter with $k_s = 10$ are depicted in Figures 3(d) and (e). From these figures, it is confirmed that the obtained filters have the equiripple characteristics in passband and the flat characteristics in stopband. From Table 2, it is confirmed that the resulting filters by the proposed method have almost the same or better characteristics compared with them by the conventional method.

4.3 Example 3

We show the design examples of the equiripple IIR filters with the flat characteristics at $\omega = 0$ and $\omega = \pi$. The design specifications are as follows: $N = 12, M = 6, \omega_p = 0.50\pi, \omega_s = 0.60\pi, \tau_d = 10.0, K_p = \{2, 6, 10\}, K_s = 2$. The total grid point $L$ is 1000 which is the sum of 500 point in the passband and 500 points in the stopband. The mag-
nitude response and group delay response of the obtained filter are shown in Figures 4(a) - (c) and the complex magnitude error and pole-zero plot of the filter with $k_p = 6$ are depicted in Figures 4(d) and (e). Moreover, the performance of each filter is shown in Table 3. In Table 3, $R_p$ is the maximum magnitude error in passband, $R_s$ is the minimum stopband attenuation, $G_{err}$ is the maximum group delay error in passband, and $P_{max}$ is the maximum pole radius. From these figures, it is confirmed that both the magnitude and group delay responses have a flat characteristic at $\omega = 0$ and $\omega = \pi$ and the equiripple characteristics are obtained in other interest region.

5 Conclusion

In this paper, we have proposed a design method for approximately linear phase IIR filters with prescribed flatness in passband or stopband, or both. The flat stopband characteristics can be realized by placing multiple zeros in the stopband. Therefore, the flat condition in stopband is preincorporated into the transfer function. With that transfer function, we formulated the approximation problem of the frequency characteristics as a QP problem with the linear matrix equality. In the proposed method, the IIR filters, which have an equiripple response and prescribed flatness, are easily obtained by solving iteratively the QP problem.

References


Table 2. Comparison between Proposed method and Ref.[2]

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<th>$R_s$</th>
<th>$G_{err}$</th>
<th>$P_{max}$</th>
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Table 3. Resulting filters in example 3

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Figure 3. Chebyshev type IIR filters in example 2.

Figure 4. Simultaneous Chebyshev type IIR filters with flatness in example 3.