EVALUATING OPTIMAL STOPPING RULES IN THE MULTIPLE BEST CHOICE PROBLEM USING THE CROSS-ENTROPY METHOD

Georgy Yu. Sofronov
Department of Statistics
Macquarie University
Sydney NSW 2109 Australia
email: georgy.sofronov@mq.edu.au

Tatiana V. Polushina
Department of Cancer Research and Molecular Medicine
Norwegian University of Science and Technology
Trondheim NO 7491 Norway
email: tatiana.polushina@ntnu.no

ABSTRACT
Best choice problems can be considered one of the most interesting problems of sequential decision analysis. Problems of this type can arise in a wide variety of fields, including psychological, economical, and ecological applications. In this study, we consider a generalization of the best choice problem when it is possible to make more than one choice. We use the Cross-Entropy method to determine the optimal stopping rules and the value of a game. We include results of numerical experiments illustrating the effectiveness of the approach. We obtain estimates of the thresholds in the optimal stopping rules and compare the accuracy of these estimates with those obtained via asymptotic approximation.

KEY WORDS
Optimal stopping rules, multiple best choice problem, secretary problem, noisy optimization, Cross-Entropy method

1 Introduction
The best choice (BC) problem or secretary problem is an important class of sequential decision problems, which arise in wide variety of fields, including psychological, economical and ecological applications. The problem can be formulated as follows. We have a known number \(N\) of objects numbered 1, 2, \ldots, \(N\), so that, say, an object numbered 1 is classified as "the best", \ldots, and an object numbered \(N\) is classified as "the worst". It is assumed that the objects arrive one by one in random order, i.e all \(N!\) permutations are equiprobable. It is clear from comparing any two of these objects which one is better, although their actual number still remain unknown. After having known each sequential object, we either accept this object (and then the choice is made), or reject it and continue observation (it is impossible to return to the rejected object).

An optimal selection rule implies that one observes and lets go an \(k^*_N - 1\) object and continues observing till the time \(\tau^*_N\), at which time the best object from all the preceding objects makes its first appearance (if the best object does not appear by the moment \(N\) we have to stop at this moment in any case). The table 1 can easily be constructed.

For large \(N\) it is approximately optimal pass up a proportion \(e^{-1} = 36.8%\) of the objects and then accept the next relatively best applicant, if any. The probability of obtaining the best object \(P^*_N\) is then approximately \(e^{-1}\).

There is an extensive literature on the BC problem. For previous work on this and related problems see [1, 2, 3, 4, 5, 6, 7, 8].

The BC problem can arise in sequential decision problems in psychological and economic applications such as selling a house, hiring a secretary or seeking a job; see [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. The BC problem can be employed to analyze some behavioral ecology problems such as sequential mate choice [22, 23, 24, 25, 26, 27] or the optimal choice of the place of foraging; see [28].

All of these works considered the case when a single object was chosen. However, it is clear that one can hire several secretaries or sell more than one house. Besides, in some species [29, 30], females sequentially mate with different males within a single mating period. Note also that an individual can sequentially choose more than one place to forage. Analysis of polygyny and polyandry, in particular, sequential polyandry [31], is one of cogent reasons to apply the BC problem with the possibility of choosing several objects.

In this paper, we consider a generalization of the BC problem when it is possible to make \(k\) choices; see [32, 33, 34]. The aim is to find stopping rules which maximize the probability of choosing \(k\) best objects. In [35, 36], the multiple best choice (MBC) problem with minimal summarized rank was considered. A problem of sequential multiple choice with given ranks was proposed in [37].

It is very difficult to obtain optimal stopping rules for this problem. However we can evaluate them by solving a combinatorial optimization problem. In this paper we present a Cross-Entropy (CE) approach to the MBC problem using Monte Carlo simulation to find optimal stopping rules.

This paper is organized as follows. Section 2 includes a statement of the MBC problem in mathematical terms. In Section 3, we explain the basic framework of the CE method. In Section 4, we develop the CE method for the MBC problem. Section 5 presents the results of two numerical experiments.
Table 1. The thresholds \( k_N^* \) and the probabilities of obtaining the best object \( P_N^* \) in the classical BC problem.

<table>
<thead>
<tr>
<th>( N )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_N^* )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>( P_N^* )</td>
<td>1.0000</td>
<td>0.5000</td>
<td>0.5000</td>
<td>0.4583</td>
<td>0.4333</td>
<td>0.4278</td>
<td>0.4098</td>
<td></td>
</tr>
</tbody>
</table>

2 The Multiple Best Choice Problem

We have a known number \( N \) of objects numbered \( 1, 2, \ldots, N \), so that, say, an object numbered 1 is classified as "the best", \ldots, and an object numbered \( N \) is classified as "the worst". It is assumed that the objects arrive one by one in random order, i.e. all \( N! \) permutations are equiprobable. It is clear from comparing any two of these objects which one is better, although their actual number still remain unknown. After having known each sequential object, we either accept this object (and then a choice of one object is made), or reject it and continue observation (it is impossible to return to the rejected object). The aim is to find stopping rules which maximize the probability of choosing \( k \) best objects.

Denote by \((a_1, a_2, \ldots, a_N)\) any permutation of numbers \((1, 2, \ldots, N)\). \((1)\) corresponds to the best object, \(N\) corresponds to the worst one. \(a_i\) is the \(m\)th object in order on quality among \((a_1, a_2, \ldots, a_i)\), we write \(y_i = m\) for all \(i = 1, 2, \ldots, N\). \(a_i\) is called absolute rank, and \(y_i\) is called relative rank. Table 2 shows absolute and relative ranks for all permutations for the case \(N = 3\).

Table 2. Absolute \((a_1, a_2, a_3)\) and relative \((y_1, y_2, y_3)\) ranks for the case \(N = 3\).

<table>
<thead>
<tr>
<th>Absolute ranks</th>
<th>Relative ranks</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 2, 3))</td>
<td>((1, 2, 3))</td>
</tr>
<tr>
<td>((1, 3, 2))</td>
<td>((1, 2, 2))</td>
</tr>
<tr>
<td>((2, 1, 3))</td>
<td>((1, 1, 3))</td>
</tr>
<tr>
<td>((2, 3, 1))</td>
<td>((1, 2, 1))</td>
</tr>
<tr>
<td>((3, 1, 2))</td>
<td>((1, 1, 2))</td>
</tr>
<tr>
<td>((3, 2, 1))</td>
<td>((1, 1, 1))</td>
</tr>
</tbody>
</table>

Remark 1 Put \(x_i = 1\) number of objects from \((a_1, a_2, \ldots, a_i) < a_i\). If our objective is to find procedure such that the expected gain \(E(x_{r_1} + \ldots + x_{r_k})\), \(k \geq 2\) is minimal, then the problem is called the MBC problem with minimal summarized rank [35, 33, 36].

Let \((i_1, \ldots, i_k)\) be any permutation of numbers \(1, 2, \ldots, k\). A rule \(\tau^* = (\tau_1^*, \ldots, \tau_k^*)\), \(1 \leq \tau_1^* < \tau_2^* < \ldots < \tau_k^* \leq N\) is an optimal rule if

\[
P\left( \bigcup_{(i_1, \ldots, i_k)} \{a_{\tau_1^*} = i_1, \ldots, a_{\tau_k^*} = i_k\} \right) = \sup_{\tau} P\left( \bigcup_{(i_1, \ldots, i_k)} \{a_{\tau_1} = i_1, \ldots, a_{\tau_k} = i_k\} \right) = P_N^*,
\]

where \(\tau = (\tau_1, \ldots, \tau_k)\).

We are interested in finding the optimal rule \(\tau^* = (\tau_1^*, \ldots, \tau_k^*)\).

By \(Z_{(m)}^{(i)} = Z_{m_1, \ldots, m_k}^{(i)}\) denote a conditional probability of event \(\{a_{m_1} = i_1, \ldots, a_{m_k} = i_k\}\) with respect to \(\sigma\)-algebra \(\mathcal{F}_{(m)}\), generated by observations \((y_1, \ldots, y_m)\), and put

\[
Z_{(m)} = \sum_{(i_1, \ldots, i_k)} Z_{(m)}^{(i)}.
\]

Using (1), we get the value of the game \(v\)

\[
P_N^* = E Z_{\tau^*} = \sup_{\tau \in S_1} E Z_{\tau} = v.
\]

Thus we reduce the BC problem of \(k\) objects to the problem of multiple stopping of the random sequence \(Z_{(m)}\); for further details see [33].

As was shown in [32, 33], the solution of this problem is the following optimal strategy: there exists a set \(\pi^* = (\pi_1^*, \ldots, \pi_k^*)\), \(1 \leq \pi_1^* < \ldots < \pi_k^* \leq N\) such that

- it is necessary to skip first \(\pi_1^* - 1\) objects, and then we stop on the first object, which is better than all precursors, or on the \((N - k + 1)\)-th object, if the best one does not appear by the moment \(N - k + 1\);
- at second time we stop on the first object, which is better than all precursors, or worse than one object (if we already have observed \(\pi_2^* - 1\) objects), if any, or, otherwise, on \((N - k + 2)\)-th object;
- the third choice should be made on the first object, which is better than all precursors, or worse than one object (if we already have observed \(\pi_3^* - 1\) objects) or worse than two objects (if we already have observed \(\pi_3^* - 1\) objects), if any, or on \((N - k + 3)\)-th object etc.

More formally,

\[
\tau_1^* = \min\{m_1 \geq \pi_1^* : y_{m_1} = 1\},
\]

\[
\tau_i^* = \min\{\min\{m_i > m_{i-1} : y_{m_i} = 1\}, \min\{m_i \geq \pi_i^* : y_{m_i} = 2\}, \ldots, \min\{m_i > m_{i-1} : m_i \geq \pi_i^*, y_{m_i} = i\}\}
\]

on the set \(F_{i-1} = \{\omega : \tau_i^* = m_1, \ldots, \tau_{i-1} = m_{i-1}\}\), \(i = 2, \ldots, k\), \(F_0 = \Omega\).
the second pairing if sperm transfer occurred in the initial pairing. In the second pairing, a female was paired with a high-crested male \((a_2 = 2)\). If sperm transfer did not occur in the second pairing, a female was paired in the final pairing with a male that had an extra-high crest \((a_3 = 1)\). This means that the relative ranks were \(y_1 = 1, y_2 = 1\) and \(y_3 = 1\). From Table 3, we obtain the optimal set \(\pi^* = (1, 3)\) for \(N = 3, k = 2\). Assuming that our model holds, from Table 5 we can see that three females out of 14 used the optimal strategy.

In the second treatment, females were paired in the initial pairing with high-crested males \((a_1 = 2)\). Females were remated in the second pairing if sperm transfer occurred in the initial pairing. In the second pairing, a female was paired with a low-crested male \((a_2 = 3)\). If sperm transfer did not occur in the second pairing, a female was paired in the final pairing with a male that had an extra-high crest \((a_3 = 1)\). In our notation, the relative ranks were \(y_1 = 1, y_2 = 2\) and \(y_3 = 1\). Table 6 shows that 10 females out of 16 used the optimal strategy.

### 3 The Cross-Entropy Method

The CE method [38] is a Monte Carlo technique for estimation and optimization. The main idea of the CE method for optimization can be stated as follows. Suppose we wish to maximize some "performance" function \(S(\mathbf{x})\) over all elements/states \(\mathbf{x}\) in some set \(\mathcal{X}\). Let us denote the maximum by \(\gamma^*\), thus

\[
\gamma^* = \max_{\mathbf{x} \in \mathcal{X}} S(\mathbf{x}).
\]

To proceed with the CE, we first randomize our deterministic problem by defining a family of probability density functions (pdfs) \(\{f(\cdot; \mathbf{u}), \mathbf{u} \in \mathcal{U}\}\) on the set \(\mathcal{X}\). Next, we associate with (3) the estimation of

\[
\ell(\gamma) = \mathbf{P}_u(S(\mathbf{X}) \geq \gamma) = \mathbf{E}_u(I_{S(\mathbf{X}) \geq \gamma}),
\]

the so-called associated stochastic problem. Here, \(\mathbf{X}\) is a random vector with pdf \(f(\cdot; \mathbf{u})\), for some \(\mathbf{u} \in \mathcal{U}\) (for example, \(\mathbf{X}\) could be a normal random vector) and \(\gamma\) is a known or unknown parameter. Note that there are in fact two possible estimation problems associated with (4). For a given \(\gamma\) we can estimate \(\ell\) or alternatively, for a given \(\ell\) we can estimate \(\gamma\), the root of (4). Let us consider the problem of estimating \(\ell\) for a certain \(\gamma\) close to \(\gamma^*\). Then, typically estimation of \(\ell\) is a non-trivial problem. The CE method solves this efficiently by making adaptive changes
to the pdf according to the Kullback-Leibler CE, thus creating a sequence \( f(\cdot, u_0), f(\cdot, u_1), f(\cdot, u_2), \ldots \) of pdfs that are “steered” in the direction of the theoretically optimal density \( f(\cdot, u^*) \) corresponding to the degenerate density at an optimal point. In fact, the CE method generates a sequence of tuples \( \{(\gamma_1, u_1)\} \), which converges quickly to a small neighborhood of the optimal tuple \( (\gamma^*, u^*) \).

In particular, starting from \( u_0 \) one proceeds as follows.

1. **Adaptive updating of \( \gamma_t \).** For a fixed \( u_{t-1} \), let \( \gamma_t \) be a \((1-\rho)\)-quantile of \( S(X) \) under \( u_{t-1} \). A simple estimator \( \hat{\gamma}_t \) of \( \gamma_t \) is

\[
\hat{\gamma}_t = S((1-\rho)N_2),
\]

where, for a random sample \( X_1, \ldots, X_{N_2} \) from \( f(\cdot; u_{t-1}) \), \( S(i) \) is the \( i \)th order statistic of the performances \( S(X_1), \ldots, S(X_{N_2}) \).

2. **Adaptive updating of \( u_t \).** For fixed \( \gamma_t \) and \( u_{t-1} \), derive \( u_t \) from the solution of the CE program

\[
\max_u D(u) = \max_u \mathbb{E}_{u_{t-1}} I_{\{S(X) \geq \hat{\gamma}_t\}} \ln f(X; u).
\]

(5)

The stochastic counterpart of (5) is as follows: for fixed \( \hat{\gamma}_t \) and \( u_{t-1} \), derive \( \tilde{u}_t \) from the following program

\[
\max_u \tilde{D}(u) = \max_u \frac{1}{N_2} \sum_{n=1}^{N_2} I_{\{S(x_n) \geq \hat{\gamma}_t\}} \ln f(x_n; u).
\]

Instead of updating the parameter vector \( u \) we use the following smoothed version

\[
\tilde{u}_t = \alpha \tilde{u}_t + (1-\alpha)u_{t-1},
\]

(6)

where \( \alpha \) is called the smoothing parameter, with \( 0.7 < \alpha \leq 1 \). Clearly, for \( \alpha = 1 \) we have our original updating rule.

To complete specification of the algorithm, one must supply values for \( N_2 \) and \( \rho \), initial parameters \( u_0 \), and a stopping criterion.

### 4 The Cross-Entropy Method For The Multiple Best Choice Problem

Recall that \( \pi^* = (\pi^*_1, \ldots, \pi^*_k) \) is the set of thresholds that we wish to find. It is difficult to obtain the set \( \pi^* \) and the value \( v \), but we can get them by simulation. So we consider the following maximization problem

\[
\max_{x \in X} \mathbb{E}\tilde{S}(x, Y),
\]

where \( X = \{x = (x_1, \ldots, x_k) : 1 \leq x_1 < \ldots < x_k \leq N\}, Y = (Y_1, \ldots, Y_N) \) is a random permutation of numbers \( 1, 2, \ldots, N \), \( \tilde{S}(x) \) is an unbiased estimator of \( \mathbb{E}\tilde{S}(x, Y) \)

\[
\tilde{S}(x) = \frac{1}{N_1} \sum_{n=1}^{N_1} I_{\{Y_{i_1} = i_1, \ldots, Y_{i_k} = i_k\}},
\]

where \( (Y_{i_1}, \ldots, Y_{i_N}) \) is the \( \text{th} \) copy of random permutation \( Y, (i_1, \ldots, i_k) \) is any permutation of numbers \( 1, 2, \ldots, k \).

Therefore we can apply the CE algorithm for noisy optimization [38].

We consider a vector of parameters \( u = \{u_l\}, \)

\[
u_l = P\{X = b_l\}, \quad l = 1, \ldots, L, \quad L = \binom{N}{k},
\]

where \( b_l = (b_{l1}, \ldots, b_{lk}), 1 \leq b_{l1} < \ldots < b_{lk} \leq N, L \) is the number of all possible combinations without repetition.
We can write the density of $x$ as

$$f(x; u) = \sum_{l=1}^{L} u_l I(x = b_l).$$

Using (3.34) [38], we get $\tilde{u}_t = \{\tilde{u}_{(t)}^i\}$:

$$\tilde{u}_{(t)}^i = \frac{\sum_{n=1}^{N_2} I(S(X_n) \geq \tilde{\gamma}_t) W_n^{(t-1)} I(X_n = b_i)}{\sum_{n=1}^{N_2} I(S(X_n) \geq \tilde{\gamma}_t) W_n^{(t-1)}}, \quad (7)$$

where $l_1$ such that $X_{n_l} = b_{l_1}, X_n = (X_{n_1}, \ldots, X_{n_k})$ is a discrete random vector from $f(x; \tilde{u}_{t-1})$. Another way of the parametrization of the vector $u$ was considered in [39].

We propose to use the following stopping criterion; see, e.g., [38], p.207. To identify stopping moment $T$, we consider the following moving average process

$$B_t(K) = \frac{1}{K} \sum_{s=t-K+1}^{t} \tilde{\gamma}_s, \quad t = s, s = 1, \ldots, s \geq K,$$

where $K$ is fixed. Define

$$C_t(K) = \frac{1}{K} \left\{ \sum_{s=t-K+1}^{t} (\tilde{\gamma}_s - B_t(K))^2 \right\}^{1/2}.$$

Define next

$$C_t^-(K, R) = \min_{j=1, \ldots, R} C_{t+j}(K)$$

and

$$C_t^+(K, R) = \max_{j=1, \ldots, R} C_{t+j}(K),$$

respectively, where $R$ is fixed.

We define stopping criterion as

$$T = \min \left\{ t : \frac{C_t^+(K, R) - C_t^-(K, R)}{C_t^+(K, R)} \leq \varepsilon \right\}, \quad (8)$$

where $K$ and $R$ are fixed and $\varepsilon$ is a small number, say $\varepsilon \leq 0.01$.

The CE algorithm for the MBC problem can be described as the following procedure.

### 4.1 Algorithm (The CE Method for the MBC Problem)

1. Choose some $\tilde{u}_0$. Set $t = 1$ (level counter).
2. Generate a sample $X_1, \ldots, X_{N_2}$ from the density $f(\cdot; u_{t-1})$ and compute the sample $(1 - \rho)$-quantile $\tilde{\gamma}_t$ of the $\tilde{S}(X)$ under $u_{t-1}$.
3. Using the same sample, update $\tilde{u}_t$ according to (7) and then smooth it out using (6).
4. If convergence condition (8) is met, then stop; otherwise set $t = t + 1$ and reiterate from step 2.

### 5 Numerical Results

**Example 3** We consider the double $(k = 2)$ BC problem with the number of objects $N = 15$. In order to find the true optimal strategy, that is, the set $\pi^* = (\pi_1^*, \pi_2^*)$, we need to generate all $15!$ possible permutations, which is a virtually impossible task. As a consequence, we do not know the true optimal pair of the thresholds but we can still look for the performance of the CE method and agreement between the numerical results and theoretical approximations. Using (2), we obtain $\pi_1^* \approx 0.229 \cdot 15 = 3.435$, $\pi_2^* \approx 0.606 \cdot 15 = 9.090$. We emphasize that these values are asymptotic approximations as $N \rightarrow \infty$.

The Figure 1 and Figure 2 show the average values of the game $v$ of 5 runs of Monte Carlo simulation with $N_1 = 1000$ and $N_1 = 5000$, respectively. We can see that the objective function (the value of the game $v$) is almost flat around its global maximum, which makes very difficult to find the optimal set of the thresholds $\pi^* = (\pi_1^*, \pi_2^*)$.

![Figure 1](image)

**Figure 1.** The value of the game $v$ depending on a set of the thresholds $\pi = (\pi_1, \pi_2)$. Note that values $v$ are arranged in ascending order

Figure 3 shows the absolute frequency histogram of the pairs obtained by 100 runs of the algorithm using $N_1 = 200$, $N_2 = 100$, $N_{last} = 5000$, $\rho = 0.1$, $\alpha = 0.7$, $K = 6$, $R = 3$, $\varepsilon = 0.01$. The CE method produces consistent estimates for the pairs.

Figure 4 plots parameters $\hat{\gamma}^{(i)}$ for pairs $(5, 9), (5, 10)$, $(6, 9)$, and the residual such that their total sum equals 1, showing rapid convergence to an optimal value. We used the CE method with $N_1 = 200$, $N_2 = 100$, $N_{last} = 5000$, $\rho = 0.1$, $\alpha = 0.7$, $K = 6$, $R = 3$, $\varepsilon = 0.0001$. This example illustrates that the CE method performs quite efficiently, even with $N_1 = 200$ out of possible 15! = 1, 307, 674, 368, 000 and the size of the sample $N_2 = 100$.

**Example 4** We consider the double $(k = 2)$ BC problem with various number of objects $N$. Table 7 displays the values of the game $v$ and the pairs obtained by asymptotic approximation (2) as well as the results of 10 runs of the
Figure 2. The value of the game $v$ depending on a pair $\pi = (\pi_1, \pi_2), \pi_1 < \pi_2$.

Figure 3. The absolute frequency histogram obtained by 100 runs of the CE method with pairs $(4, 8), (6, 11), (4, 10), (5, 9)$ emphasized.

Figure 4. The curves of parameters $\hat{u}^{(t)}$ for pairs $(5, 9), (5, 10), (6, 9)$, and the residual.

Figure 5. The curve of the value of the game $v$ depending on the number of objects $N$. Notice that according to (2) the asymptotic value of the game equals 0.225. Figure 6 shows average ratios $\pi_1^*/N, \pi_2^*/N$ obtained by 10 runs of the CE method with $N_1 = 500, N_2 = 100, N_{last} = 500, \rho = 0.1, \alpha = 0.7, K = 6, R = 3, \varepsilon = 0.01$.

CE method with $N_1 = 500, N_2 = 100, N_{last} = 10000, \rho = 0.1, \alpha = 0.7, K = 6, R = 3, \varepsilon = 0.01$.

Figure 5 plots average values of the game $v$ and the standard deviations depending on the number of objects $N$. Notice that according to (2) the asymptotic value of the game equals 0.225. Figure 6 shows average ratios $\pi_1^*/N, \pi_2^*/N$ obtained by 10 runs of the CE method with $N_1 = 500, N_2 = 100, N_{last} = 500, \rho = 0.1, \alpha = 0.7, K = 6, R = 3, \varepsilon = 0.01$. Recall that

$$\lim_{N \to \infty} \frac{\pi_1^*}{N} = 0.229, \quad \lim_{N \to \infty} \frac{\pi_2^*}{N} = 0.606.$$

This example illustrates that the CE method performs quite well notwithstanding the parameters $N_1$ and $N_2$ are very small comparing to the total number of permutations and the number of different sets $\pi = (\pi_1, \pi_2)$, respectively.

6 Conclusion

In this paper, we have proposed a new method based on the CE approach for the evaluation of optimal stopping rules in the MBC problem. The examples reported in the paper show the effectiveness of this method. The methodology can also be extended to other generalizations of the BC problem.
Table 7. The values of game $v$ and the pairs $\pi = (\pi_1, \pi_2)$ obtained by the asymptotic approximation and the CE method depending on the number of objects $N$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Asymptotic $\pi$</th>
<th>CE method $\pi$</th>
<th>min $v$</th>
<th>mean $v$</th>
<th>max $v$</th>
<th>std $v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>(2, 6)</td>
<td>(4, 8)</td>
<td>0.2407</td>
<td>0.2421</td>
<td>0.2436</td>
<td>0.0010</td>
</tr>
<tr>
<td>20</td>
<td>(5, 12)</td>
<td>(6, 17)</td>
<td>0.2235</td>
<td>0.2248</td>
<td>0.2258</td>
<td>0.0008</td>
</tr>
<tr>
<td>30</td>
<td>(7, 18)</td>
<td>(8, 23)</td>
<td>0.2169</td>
<td>0.2192</td>
<td>0.2206</td>
<td>0.0011</td>
</tr>
<tr>
<td>40</td>
<td>(9, 24)</td>
<td>(11, 30)</td>
<td>0.2127</td>
<td>0.2156</td>
<td>0.2175</td>
<td>0.0014</td>
</tr>
<tr>
<td>50</td>
<td>(11, 30)</td>
<td>(14, 37)</td>
<td>0.2112</td>
<td>0.2142</td>
<td>0.2198</td>
<td>0.0011</td>
</tr>
</tbody>
</table>

Figure 6. The curves of the ratios $\pi_1^*/N$, $\pi_2^*/N$ depending on the number of objects $N$.

Acknowledgements

The authors are grateful to Prof. D. P. Kroese for comments on the manuscript. Tatiana Polushina would like to acknowledge ERCIM “Alain Bensoussan” Fellowship Programme, which is supported by the Marie Curie Co-funding of Regional, National and International Programmes (COFUND) of the European Commission.

Table 8. The total number of permutations and the number of different pairs $\pi = (\pi_1, \pi_2)$ depending on $N$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Permutations</th>
<th>Pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$3.6 \times 10^6$</td>
<td>45</td>
</tr>
<tr>
<td>20</td>
<td>$2.4 \times 10^{18}$</td>
<td>190</td>
</tr>
<tr>
<td>30</td>
<td>$2.6 \times 10^{32}$</td>
<td>435</td>
</tr>
<tr>
<td>40</td>
<td>$8.1 \times 10^{37}$</td>
<td>780</td>
</tr>
<tr>
<td>50</td>
<td>$3.0 \times 10^{44}$</td>
<td>1225</td>
</tr>
</tbody>
</table>

References


