COOPERATIVE GLOBAL ROBUST OUTPUT REGULATION FOR NONLINEAR MULTI-AGENT SYSTEMS IN OUTPUT FEEDBACK FORM

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ABSTRACT
In this paper we consider the global robust output regulation problem for a class of nonlinear multi-agent systems by distributed output feedback control. We first show that the problem can be converted into the global stabilization problem of a class of multi-input, multi-output nonlinear systems called augmented system. Then we further show that, under a set of standard assumptions, the augmented system can be globally stabilized by a distributed output feedback control law. Finally, we apply our approach to solve a leader-following synchronization problem for a group of Lorenz multi-agent systems.

KEY WORDS
nonlinear, output regulation, multi-agent system

1 Introduction

Recently, the cooperative robust output regulation problem for linear multi-agent systems was studied in [9, 11, 12]. The problem can be viewed as a generalization of the leader-following consensus/synchronization problem because it will not only address the issue of asymptotic tracking but also address such issues as disturbance rejection, robustness with respect to parameter uncertainties, etc. The same problem was also studied for a class of nonlinear systems in [6]. However, only a local solution was given in [6]. In this paper, we will further consider the cooperative output regulation problem for the following class of nonlinear systems:

\begin{align}
\dot{z}_i &= f_i(z_i, y_i, v, w) \\
\dot{y}_i &= b_i(v, w)u_i + g_i(z_i, y_i, v, w) \\
e_i &= y_i - q(v, w), \quad i = 1, \ldots, N
\end{align}

where, for \( i = 1, \ldots, N \), \((z_i, y_i) \in \mathbb{R}^n \times \mathbb{R}\) is the state, \( u_i \in \mathbb{R} \) is the input, \( e_i \in \mathbb{R} \) is the error output, \( w \in \mathbb{W} \subset \mathbb{R}^{n_w} \) is an uncertain parameter vector with \( \mathbb{W} \) an arbitrarily prescribed subset of \( \mathbb{R}^{n_w} \), and \( v(t) \in \mathbb{R}^{n_v} \) is an exogenous signal presenting both reference input and disturbance. It is assumed that \( v(t) \) is generated by a linear system of the following form

\[ \dot{v} = Sv, \quad y_0 = q(v, w) \]  

It is assumed that all functions in (1) are globally defined, sufficiently smooth, and satisfy \( f_i(0, 0, 0, w) = 0 \), \( g_i(0, 0, 0, w) = 0 \), and \( q(0, w) = 0 \) for all \( w \in \mathbb{W} \).

The system composed of (1) and (2) can be viewed as a multi-agent system of \((N+1)\) agents with (2) as the leader and the \( N \) subsystems of (1) as followers. With respect to the system composed of (1) and (2), we can define a digraph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) where \( \mathcal{V} = \{0, 1, \ldots, N\} \) with 0 associated with the leader system and with \( i, i = 1, \ldots, N \), associated with the N followers, respectively, and \((j, i) \in \mathcal{E}, j = 0, 1, \ldots, N\) and \( i = 1, \ldots, N \), if and only if the control \( u_i \) can make use of \( y_j \) for feedback control. Thus our control law is of the following form:

\[ u_i = \hat{k}_i(\eta_i, y_i - y_j, j \in \mathcal{N}_i), \quad \dot{\eta}_i = \hat{g}_i(\eta_i, y_i - y_j, j \in \mathcal{N}_i) \]

where \( \mathcal{N}_i \) is the neighbor set of the node \( i \), \( \hat{k}_i \) and \( \hat{g}_i \) are sufficiently smooth functions vanishing at the origin, and \( \eta_i \in \mathbb{R}^{n_{\eta_i}} \) with \( n_{\eta_i} \) to be defined later. A control law of the form (3) is called a distributed dynamic output feedback control law because the control of each subsystem can only take the information of its neighbors and itself for control.

Define a subgraph \( \mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i) \) of \( \mathcal{G} \) where \( \mathcal{V}_i = \{1, \ldots, N_i\} \), \( \mathcal{E}_i \subseteq \mathcal{V} \times \mathcal{V}_i \) is obtained from \( \mathcal{E} \) by removing all edges between the node 0 and the nodes in \( \mathcal{V}_i \). For \( i = 1, \ldots, N \), let \( \mathcal{N}_i = \mathcal{N}_i(t) \cap \mathcal{V}_i \). It can be seen that, for \( i = 1, \ldots, N, \mathcal{N}_i \) is the neighbor set of the node \( i \) with respect to \( \mathcal{V}_i \).

We call the composition of (1) and (3) as the overall closed-loop system which can be put in the following form

\[ \dot{x}_c = f_c(x_c, v, w) \]

where \( x_c = \text{col}(z_1, y_1, \eta_1, \ldots, z_N, y_N, \eta_N) \in \mathbb{R}^{n_{x_c}} \) for some integer \( n_{x_c} \). Then \( f_c \) is sufficiently smooth satisfying \( f_c(0, 0, w) = 0 \) for all \( w \in \mathbb{W} \). Then we can describe our problem as follows:

\[^1\text{See [10] for a summary of digraph.}\]
\[^2\text{col}(a_1, \ldots, a_s) = [a_1^T, \ldots, a_s^T]^T \] where \( a_i, i = 1, \ldots, s \) are any column vectors.
Definition 1.1 Given the multi-agent system (1), the exosystem (2), the corresponding digraph \( \mathcal{G} \), and any compact subsets \( \mathcal{V} \subset \mathbb{R}^{n_v} \) and \( \mathcal{W} \subset \mathbb{R}^{n_w} \) which contain \( v = 0 \) and \( w = 0 \), respectively, find a control law of the form (3) such that, for any \( v(0) \in \mathcal{V} \), \( w(0) \in \mathcal{W} \), the trajectory of the closed-loop system (4) starting from any initial state \( \tilde{x}(0) \) exists and is bounded for all \( t \geq 0 \), and \( \lim_{t \to \infty} e(t) = 0 \) with \( e = \text{col}(e_1, \ldots, e_N) \).

It can be seen that, for the special case where \( N = 1 \), the above problem is the robust output regulation problem for output feedback systems as studied in [13]. The current problem is more challenging and interesting than that of [13] in at least two ways. First, the system in [13] is a single-input, single-output system. It can be converted into a global stabilization problem of an augmented system through the employment of an internal model. The augmented system is still a single-input, single-output system whose stabilization problem can be handled by established techniques for global stabilization. In contrast, for a multi-agent system, the augmented system is a multi-input, multi-output nonlinear system and we have to develop techniques that apply to multi-input, multi-output nonlinear systems. Second, due to the communication constraint described by the communication graph \( \mathcal{G} \), we have to limit ourselves to the distributed control law as described in (3) to solve the stabilization problem for the augmented system.

The rest of this paper is organized as follows. In Section 2, we will present the preliminaries for our problem. In Section 3, we will present our main result. In Section 4, we will apply our approach to solve a leader-following synchronization problem for a group of Lorenz systems. Finally, we close this paper in Section 5 with some concluding remarks.

2 Preliminaries

By the general framework for handling the output regulation problem for nonlinear systems described in [4], the first step of our approach is to find an internal model for (1) to form an augmented system. This step for the special case where \( N = 1 \) was conducted in [13]. Here we will generalize the procedure in [13] to the general case with any \( N > 1 \). For this purpose, we need to make some standard assumptions as follows:

Assumption 2.1 The exosystem is neutrally stable, i.e., all the eigenvalues of \( S \) are semi-simple with zero real parts.

Assumption 2.2 \( |b_i(v, w)| > 0, \quad i = 1, \ldots, N, \) for all \( v \in \mathbb{R}^{n_v} \) and all \( w \in \mathbb{R}^{n_w} \).

Remark 2.1 Without loss of generality, we can assume \( b_i(v, w) > 0, \quad i = 1, \ldots, N, \) for all \( v \in \mathbb{R}^{n_v} \) and all \( w \in \mathbb{R}^{n_w} \). In this case, for any known compact subsets \( \mathcal{V} \) and \( \mathcal{W} \), there exist some known positive numbers \( b_{\max} \) and \( b_{\min} \) such that, \( i = 1, \ldots, N, \) \( b_{\min} \leq b_i(v, w) \leq b_{\max} \) for all \( v \in \mathbb{R}^{n_v} \) and all \( w \in \mathbb{R}^{n_w} \).

Assumption 2.3 There exist globally defined smooth functions \( z_i : \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \to \mathbb{R}^n \) with \( z_i(0, w) = 0 \) such that
\[
\frac{\partial z_i(v, w)}{\partial v} S v = f_i(z_i(v, w), q(v, w), v, w) \quad (5)
\]
for all \( (v, w) \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_w} \).

Under Assumption 2.3, let \( y_i(v, w) = q(v, w) \) and
\[
u_i(v, w) = b_i^{-1}(\frac{\partial q(v, w)}{\partial v} S v - g_i(z_i(v, w), q(v, w), v, w)) \quad (6)
\]
Then, \( z_i(v, w), y_i(v, w) \) and \( \nu_i(v, w) \) are the solutions for the regulator equations associated with Eqs. (1) and (2).

Assumption 2.4 \( u_i(v, w), \quad i = 1, \ldots, N, \) are polynomials in \( v \) with coefficients depending on \( w \).

Remark 2.2 As remarked in [13], under Assumption 2.4, there exists \( i \) such that \( u_i(v, w) \) satisfy, for all trajectories \( v(t) \) of the exosystem and all \( w \in \mathcal{W} \)
\[
d^m u_i \quad dv = a_1 u_i + a_2 \frac{du_i}{dt} + \cdots + a_s u_i \frac{dt^{s-1}}{dt} \quad (7)
\]
where \( a_{11}, a_{21}, \ldots, a_{ss} \) are real scalars such that all the roots of the polynomial \( P_i(\lambda) = \lambda^s - a_{11} \lambda^{s-1} - \cdots - a_{ss} \) are distinct with zero real parts [3].

Let \( \tau_i(v, w) = \text{col}(u_i, \ldots, u_i^s) \) and \( \Phi_i \) and \( \Gamma_i \) satisfy the following equations:
\[
\frac{\partial \tau_i(v, w)}{\partial v} S v = \Phi_i \tau_i(v, w), \quad u_i(v, w) = \Gamma_i \tau_i(v, w) \quad (9)
\]
System (9) can be used to generate the steady-state input \( u_i(v, w) \), and thus it is called a steady-state generator with output \( u_i \) [4]. Since \( (\Gamma_i, \Phi_i) \) is observable and the eigenvalues of \( \Phi_i \) have zero real parts, for any controllable pair \( (M_i, N_i) \), where \( M_i \in \mathbb{R}^{s \times s_i} \) is a Hurwitz matrix and \( N_i \in \mathbb{R}^{s_i \times 1} \) is a column vector, there is a unique nonsingular matrix \( T_i \) satisfying the Sylvester equation
\[
T_i \Phi_i - M_i \Gamma_i = N_i \Gamma_i \quad (10)
\]
Let \( \theta_i(v, w) = T_i \tau_i(v, w) \), which satisfies \( \dot{\theta}_i = (M_i + N_i \Psi_i) \theta_i \) and \( u_i(v, w) = \Psi_i \theta_i \), where \( \Psi_i = \Gamma_i T_i^{-1} \). Then we can define a dynamic compensator as follows [7]:
\[
\dot{\eta}_i = M_i \eta_i + N_i u_i \quad (11)
\]
which is called the internal model with output \( u_i \) in the sense of Definition 3.4 in [4].

Attaching the internal model (11) to (1) and performing the following coordinate and input transformation:
\[
\tilde{z}_i = z_i - z_i(v, w), \quad \tilde{\eta}_i = \eta_i - \theta_i(v, w) - N_i b_i^{-1} e_i \quad (12)
\]
\[
e_i = y_i - q(v, w), \quad \tilde{u}_i = u_i - \Psi_i \eta_i, \quad i = 1, \ldots, N
\]
leads to the following so-called augmented system
\begin{equation}
\dot{z}_i = \tilde{f}_i(\bar{z}_i, e_i, \mu)
\end{equation}
\begin{equation}
\dot{\eta}_i = M_i \bar{\eta}_i + M_i N_i b_i^{-1} e_i - N_i b_i^{-1} \eta_i (\bar{z}_i, e_i, \mu)
- N_i \frac{\partial b_i^{-1}(v, w)}{\partial v} S u e_i
\end{equation}
\begin{equation}
\dot{e}_i = \tilde{g}_i(\bar{z}_i, e_i, \mu) + b_i \Psi_i \bar{\eta}_i + \Psi_i N_i e_i + b_i \tilde{u}_i
\end{equation}

for some class $K$ functions $\Psi_i(\cdot)$ such that, for any $\mu \in \Omega$ along the trajectory of the subsystem $\dot{z}_i = f_i(\bar{z}_i, e_i, \mu)$, $V_{z_i}(\bar{z}_i) \leq -\alpha_i(\|\bar{z}_i\|) + \gamma_i(e_i)$, where $\alpha_i(\cdot)$ is some known class $K$ function satisfying $\lim_{s \to 0^+} \sup \alpha_i(s^2)/s < \infty$, and $\gamma_i(\cdot)$ is some known smooth positive definite function.

**Remark 3.1** This assumption is quite standard in the literature of the global robust stabilization and output regulation [5], [13]. It guarantees that the subsystem $\dot{z}_i = f_i(\bar{z}_i, e_i, \mu)$ is input-to-state stable [8]. Under this assumption, by Lemma 3.1 in [13], there exists a $C^1$ function $V_i(Z_i)$ satisfying $\|\dot{V}_i(Z_i)\| \leq V_i(Z_i) - \|e_i\| + \psi_i(e_i)$, for some class $K$ function $\psi_i(\cdot)$ such that, for all $\mu \in \Omega$, along the trajectory of $Z_i$ subsystem in (19),
\begin{equation}
\dot{V}_i(Z_i) \leq -\|e_i\|^2 + \psi_i(e_i)
\end{equation}

for some known smooth positive definite function $\psi_i(e_i)$.

**Assumption 3.1** For any compact subset $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{n_w}$, there exists a $C^1$ function $\bar{V}_i(Z_i)$ satisfying $\Omega_{i1}(\|\bar{z}_i\|) \leq V_i(Z_i) \leq \|z_i\|$, for some class $K$ functions $\Omega_{i1}(\cdot)$ and $\Omega_{i2}(\cdot)$ such that, for any $\mu \in \Omega$, along the trajectory of the subsystem $\dot{z}_i = f_i(\bar{z}_i, e_i, \mu)$, $V_{z_i}(\bar{z}_i) \leq -\alpha_i(\|\bar{z}_i\|) + \gamma_i(e_i)$, where $\alpha_i(\cdot)$ is some known class $K$ function satisfying $\lim_{s \to 0^+} \sup \alpha_i(s^2)/s < \infty$, and $\gamma_i(\cdot)$ is some known smooth positive definite function.

**Assumption 3.2** Every node $i = 1, \ldots, N$ is reachable from the node 0 in the digraph $\mathcal{G}$, and $\mathcal{G}$ is an undirected graph.

**Remark 3.2** Let $\Delta$ be an $N \times N$ nonnegative diagonal matrix whose $i$th diagonal element is $\delta_i$, $i = 1, \ldots, N$. Then $L \triangleq \begin{bmatrix} 0 & 0_{1 \times N} \\
-\Delta & H \end{bmatrix}$, where $H = [h_{ij}]_{i,j=1}^N$ with $h_{ii} = \sum_{j=1}^N c_{ij}$ and $h_{ij} = -c_{ij}$, $0_{m \times n}$ denoting the zero matrix in $\mathbb{R}^{m \times n}$, is a Laplacian of $\mathcal{G}$. So $H \mathbb{1}_N = \Delta \mathbb{1}_N$, where $\mathbb{1}_N$ denotes an $N \times 1$ column vector whose elements are all 1. Moreover, by Lemma 4 in [2], all the eigenvalues of $H$ have positive real parts if and only if Assumption 3.2 is satisfied. Since $\mathcal{G}$ is an undirected graph, $H$ is also symmetric.

**Lemma 3.1** Under Assumptions 2.1-2.4, 3.1 and 3.2, the global stabilization problem of system (13) can be solved by the distributed output feedback control law of the form
\begin{equation}
\bar{u}_i = -\rho_i(e_i) e_i, \quad i = 1, \ldots, N
\end{equation}
where $\rho_i(\cdot)$, $i = 1, \ldots, N$, are some sufficiently smooth positive definite functions to be specified in the proof of this lemma.

**Proof:** By the changing supply rate technique [8], given any smooth function $\bar{V}_i(Z_i) > 0$, there exists a $C^1$ function $\tilde{V}_i(Z_i)$ satisfying $\Omega_0(\|\bar{z}_i\|) \leq \bar{V}_i(Z_i) \leq \Omega_0^2(\|\bar{z}_i\|)$ for some class $K$ functions $\Omega_0(\cdot)$ and $\Omega_0^2(\cdot)$, such that, for all $\mu \in \Omega$, along the trajectory of $Z_i$ subsystem of (19),
\begin{equation}
\tilde{V}_i(Z_i) \leq -\bar{V}_i(Z_i) \|\bar{z}_i\|^2 + \bar{V}_i(Z_i) |e_i|^2
\end{equation}

3 Main Result

In this section, we will focus on globally stabilizing the augmented system (13) by a control law of the form (17). For this purpose, we need two more assumptions as follows
where \( \pi_i(...) \), \( i = 1, \ldots, N \), are some known smooth positive definite functions.

Next, let \( Z = \text{col}(Z_1, \ldots, Z_N) \) and
\[
\tilde{g}_i(Z_i, e_i, \mu) = \frac{1}{2}e^T H e - \psi_i(e_i) - \pi_i(e_i) e_i^2 \geq 0,
\]
\[
\tilde{g}(Z, e, \mu) = \text{col}(\tilde{g}_1(Z_1, e_1, \mu), \ldots, \tilde{g}_N(Z_N, e_N, \mu))
\]
Since \( \tilde{g}_i(Z_i, e_i, \mu) \), \( i = 1, \ldots, N \), are all smooth and satisfy \( \tilde{g}_i(0, 0, 0, \mu) = 0 \), by Lemma 7.8 of [3] again, there exist some smooth functions \( \delta_i(Z_i) \geq 1 \) and \( \chi_i(e_i) \geq 1 \), such that, for all \( Z_i \in \mathbb{R}^{n_i} \), \( e_i \in \mathbb{R} \) and \( \mu \in \Omega \),
\[
\|\tilde{g}_i(Z_i, e_i, \mu)\| \leq \delta_i(Z_i)\|Z_i\|^2 + \chi_i(e_i) e_i^2
\]
(23)
Let \( e_i = \text{col}(e_{i1}, \ldots, e_{iN}) \). Then \( e = H^{-1} e_i \). Let \( V_e = \frac{1}{2}e^T H e \). Then the derivative of \( V_e \) along the subsystems \( \dot{e}_i = \tilde{g}_i(Z_i, e_i, \mu) - b_i \rho_i(e_i) e_i \), \( i = 1, \ldots, N \), satisfies
\[
\dot{V}_e = e^T H \dot{e} = e^T H (\tilde{g}_i(Z_i, e_i, \mu) - b_i \rho_i(e_i) e_i) = \sum_{i=1}^{N} -b_i \rho_i(e_i) e_i^2 + \frac{1}{2}e^T H e = \sum_{i=1}^{N} -b_i \rho_i(e_i) e_i^2 + \frac{1}{2}e^T H e \]
(24)
with \( \epsilon > 0 \). Let \( V(Z_e) = \sum_{i=1}^{N} \tilde{V}_i(Z_i) \). Finally, let
\[
U(Z, e) = V_Z + V_e.
\]
Then the derivative of \( U \) along the trajectory of the closed-loop system satisfies
\[
\dot{U} \leq - \sum_{i=1}^{N} b_i \rho_i(e_i) e_i^2 + \frac{1}{2} \sum_{i=1}^{N} \tilde{V}_i(Z_i) + \frac{1}{2} \sum_{i=1}^{N} \delta_i(Z_i)\|Z_i\|^2 + \frac{1}{2} \chi_i(e_i) e_i^2
\]
(25)
with \( \rho(e) = \sum_{i=1}^{N} \frac{1}{2} \rho_i(e_i) e_i^2 + \frac{1}{2} \chi_i(e_i) e_i^2 \).

\[
\text{By Lemma 7.8 of [3], there exist known smooth positive definite functions } \tilde{\rho}_i(e_i), \text{ such that } \tilde{\rho}(e) = \tilde{\rho}(H^{-1} e_i) \leq \sum_{i=1}^{N} \tilde{\rho}_i(e_i) e_i^2.
\]

Let \( \rho_i(e_i) \geq b_{i\text{min}}(\rho_i(e_i) + \epsilon_i) \) and \( \delta_i(Z_i) \leq \tilde{\delta}_i(Z_i) + \epsilon_i \) with \( \epsilon_1, \epsilon_2 > 0 \). Then \( \dot{U} \leq -\epsilon_i \|Z_i\|^2 - \epsilon_2 \|e_i\|^2 \). Thus under the output feedback control law (21), the equilibrium of the closed-loop system is uniformly globally asymptotically stable. Thus the proof is completed.

Lemma 3.1 leads to our main result as follows.

**Theorem 3.1** Under Assumptions 2.1-2.4, 3.1 and 3.2, the cooperative global robust output regulation problem of system (1) can be solved by the distributed output feedback control law of the form
\[
u_i = -\rho_i(e_i) e_i + \psi_i \eta_i, \quad i = 1, \ldots, N
\]
\[
\eta_i = M_i \eta_i + N_i u_i
\]
(26)

### 4 An Example

Consider the following multi-agent system:
\[
\begin{align*}
\dot{x}_{1i} &= -L_{1i} x_{1i} + L_{1i} x_{2i} \\
\dot{x}_{2i} &= b_i u_i + L_{3i} x_{1i} - x_{2i} - x_{1i} x_{3i} \\
\dot{x}_{3i} &= L_{2i} x_{3i} + x_{1i} x_{2i}
\end{align*}
\]
(27)
where \( b_i = 1, i = 1, \ldots, N \). The exosystem is
\[
\begin{bmatrix}
\dot{v}_1 \\
\dot{v}_2 
\end{bmatrix} =
S
\begin{bmatrix}
v_1 \\
v_2 
\end{bmatrix} =
\begin{bmatrix}
0 & \omega \\
-\omega & 0
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 
\end{bmatrix}
\]
(28)
with \( \omega = 1 \). For each \( i \), system (28) is called controlled Lorenz system [1], [13].

By letting \( z_{1i}, z_{2i}, y_{1i} = (x_{1i}, x_{3i}, x_{2i}) \), we can put the system (27) in the standard form (1) as follows:
\[
\begin{align*}
\dot{z}_{1i} &= -L_{1i} z_{1i} + L_{1i} y_{1i} \\
\dot{z}_{2i} &= L_{2i} z_{2i} + y_{1i} \\
\dot{y}_{1i} &= b_i u_i + L_{3i} z_{1i} - y_{1i} - z_{1i} z_{2i} \\
e_i &= y_{1i} - v_i
\end{align*}
\]
(29)
For the special case where \( N = 1 \), the output regulation problem for this system was studied by a decentralized control law in [13]. Here we assume \( N = 3 \) and the interconnection among various subsystems is determined by Figure 1. We will design a distributed control law to solve our problem. To make our problem more interesting, as in [13], we allow the parameter vector \( L_i = (L_{1i}, L_{2i}, L_{3i}) \) to undergo some perturbation. To be more specific, let the normal value of \( L_i \) be \( (10, -\frac{8}{3}, 28) \), \( i = 1, 2, 3 \). Then \( L_i = (10, -\frac{8}{3}, 28) + (w_{1i}, w_{2i}, w_{3i}) \), where \( (w_{1i}, w_{2i}, w_{3i}) \) represents the uncertainty of \( L_i \) for \( i = 1, 2, 3 \). Let \( w_i = \text{col}(w_{1i}, w_{2i}, w_{3i}) \), \( i = 1, 2, 3 \), and \( w = \text{col}(w_1, w_2, w_3) \). Define \( W = \{w|w \in \mathbb{R}^9, ||w_i|| \leq 1, i = 1, 2, 3\} \), and \( V = \{v(t)|||v(t)|| \leq 1\} \).

Using the result of [13], it can be directly inferred that, for each \( i = 1, 2, 3 \), the composite system (28) and (29) satisfies Assumptions 2.1-2.4. In fact, for the sake of self-containment, we can get \( y_i(v, w) = v_i \) from the last equation of (29). Substituting \( y_i(v, w) = v_i \) into the equations of (29) yields
\[
\begin{align*}
z_{1i}(v, w) &= r_{11i} v_1 + r_{21i} v_2 \\
z_{2i}(v, w) &= r_{12i} v_1^2 + r_{22i} v_2^2 + r_{32i} v_1 v_2 \\
u_i(v, w) &= r_{31i} v_1 + r_{32i} v_2 + r_{33i} v_1^3 + r_{34i} v_2^3 \\
&+ r_{35i} v_1^2 v_2 + r_{36i} v_1 v_2^2
\end{align*}
\]
(30)
with

\[ r_{11i}(w) = \frac{L_{11i}}{\omega^2 + L_{11i}}, \quad r_{12i}(w) = \frac{L_{11i} \omega}{\omega^2 + L_{11i}} \]

\[ r_{21i}(w) = -\frac{\omega r_{21i} + r_{11i}}{L_{2i}}, \quad r_{22i}(w) = \frac{\omega r_{21i}}{L_{2i}} \]

\[ r_{31i}(w) = -b_i^{-1}(L_{3i} R_{11i} - 1), \quad r_{32i}(w) = b_i^{-1}(\omega - L_{3i} R_{12i}) \]

\[ r_{33i}(w) = b_i^{-1} r_{11i} r_{21i}, \quad r_{34i}(w) = b_i^{-1} r_{12i} r_{22i} \]

\[ r_{35i}(w) = b_i^{-1} (r_{11i} r_{21i} + r_{11i} r_{23i}) \]

\[ r_{36i}(w) = b_i^{-1} (r_{11i} r_{22i} + r_{12i} r_{23i}) \]

It can be further verified that

\[ \frac{d^4 u_i(v, w)}{d t^4} + 9 \omega^4 u_i(v, w) + 10 \omega^2 \frac{d^2 u_i(v, w)}{d t^2} = 0 \]  

(32)

From Figure 1, we can directly see the satisfaction of Assumption 3.1. We only need to further verify Assumption 3.2. For this purpose, note that either from (32) or from [13], we can obtain the the steady-state generator (9) as follows

\[ \tau_i(v, w) = \text{col}(u_i, u_i, u_i^{(2)}, u_i^{(3)}) \]  

(33)

\[ \Phi_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 \omega^4 & 0 & -10 \omega^2 & 0 \end{bmatrix}, \quad \Gamma_i = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]  

(34)

which leads to the internal model (11) with

\[ M_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -12 & -13 & -6 \end{bmatrix}, \quad N_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \]

Solving the Sylvester equation \( T_i \Phi_i - M_i T_i^T = N_i \Gamma_i \) gives \( \Psi_i = \Gamma_i T_i^{-1} = [4 - 9 \omega^4, 12, 13 - 10 \omega^2, 6] \).

Performing the coordinate transformation (12) yields the augmented system (13) as follows:

\[ \dot{z}_{1i} = -L_{1i} z_{1i} + L_{1i} e_i \]

\[ \dot{z}_{2i} = L_{2i} z_{2i} + (z_{1i} + z_{1i})(e_i + v_i) - z_{1i} v_i \]

\[ \dot{\eta}_i = M_i \dot{\eta}_i + M_i N_i b_i^{-1} e_i - N_i b_i^{-1} g_i(z_{1i}, z_{1i}, e_i, \mu) \]

\[ \dot{e}_i = g_i(z_{1i}, z_{1i}, e_i, \mu) + b_i \eta_i + \Psi_i N_i e_i + b_i u_i \]

(35)

with \( g_i(z_{1i}, z_{1i}, e_i, \mu) = L_{3i} z_{1i} - e_i - z_{1i} z_{2i} - z_{1i} z_{2i} - z_{1i} z_{2i} \).

For the \((z_{1i}, z_{2i})\)-subsystem in (35), let

\[ V_{z_i} = \frac{h_1}{2} z_{1i}^2 + \frac{h_1}{4} z_{1i}^4 + \frac{h_1}{8} z_{1i}^6 + \frac{h_2}{2} z_{2i}^2 + \frac{h_2}{4} z_{2i}^4 \]  

(36)

The time derivative of \((z_{1i}, z_{2i})\)-subsystem is given by

\[ \dot{V}_{z_i} = h_1 \dot{z}_{1i} \dot{z}_{1i} + h_1 \dot{z}_{1i}^2 \dot{z}_{1i} + h_1 \dot{z}_{1i}^4 \dot{z}_{1i} + h_2 \dot{z}_{2i} \dot{z}_{2i} + h_2 \dot{z}_{2i}^2 \dot{z}_{2i} \]

\[ = -h_1 L_{1i} z_{1i}^2 + h_1 L_{1i} \dot{z}_{1i} e_i - h_1 L_{1i} \dot{z}_{1i} e_i + h_1 L_{1i} z_{1i}^2 e_i \]

\[ - h_1 L_{1i} z_{1i}^4 e_i + h_1 L_{1i} z_{1i}^2 e_i + L_{2i} \dot{z}_{2i}^2 z_{2i} + h_2 z_{2i} \dot{z}_{2i} e_i + h_2 \dot{z}_{2i} z_{1i} e_i + h_2 \dot{z}_{2i} z_{1i} e_i + h_2 \dot{z}_{2i} z_{1i} e_i \]

(37)

By completing the square in (37), we can further obtain

\[ \dot{V}_{z_i} \leq -l_1 z_{1i}^2 - l_2 z_{2i}^2 - l_4 z_{1i}^2 - l_5 z_{2i}^2 + l_6 e_i^2 + l_7 e_i^4 + l_8 e_i^8 \]  

(38)

where

\[ l_1 = h_1 L_{1i} - \frac{0.01}{2} \frac{v_i^2 h_2}{2}, \quad l_2 = h_1 L_{1i} - 1, \quad l_4 = -L_{2i} h_2 - \frac{3}{2} \]

\[ l_5 = -h_2 L_{2i} - \frac{9}{4}, \quad l_6 = \frac{\frac{h_2^2}{2}}{2} + \frac{h_2^2 \dot{z}_{2i}^2}{2} \]

\[ l_7 = \frac{h_1^4 L_{1i}^4}{4} + \frac{h_2^2}{2} + \frac{h_2^2 \dot{z}_{1i}^2}{4}, \quad l_8 = \frac{h_8^2}{8} + \frac{h_8^2}{8} \]  

(39)

It can be seen that for proper \( h_1 > 0 \) and \( h_2 > 0 \), \( l_1, \ldots, l_8 > 0 \). Thus Assumption 3.1 is also satisfied. By Theorem 3.1, there exists a distributed output feedback control law of the form (26) to solve the cooperative robust output regulation problem for this system. In fact, using the approach detailed in the proof of Lemma 3.1, we can first construct a control law of the form (21) with \( \rho_i(e_{vi}) = 4.187 \times 10^6 e_{vi} + 4714 L_{1i}^2 e_{vi} + 1778 \) that globally stabilizes the augmented system (35). Then, a control law of the form (26) will solve the cooperative output regulation problem for this example.

The performance of this control law is evaluated by computer simulation with the following initial conditions of the closed-loop system

\[ [z_1(0), y_1(0)] = [0 \ 0 \ 0.5]^T \]

\[ [z_2(0), y_2(0)] = [1 \ 2 \ 1.5]^T \]

\[ [z_3(0), y_3(0)] = [1 \ 2 \ 0.8]^T \]

\[ v(0) = [1 \ 0]^T, \quad \eta_i(0) = [0 \ 0 \ 0 \ 0]^T \]

\[ \eta_2(0) = [0 \ 0 \ 6 \ 0]^T, \quad \eta_3(0) = [0 \ 6 \ 0 \ 6]^T \]

and the following values of the uncertain parameters \( w_i = \text{col}(0.1i, 0.1i, 0.1i), i = 1, 2, 3 \).

Figures 2-4 show the tracking error and state of each follower, respectively. It can be seen that the tracking error of all subsystems approach the origin asymptotically.

5 Conclusion

In this paper, we have studied the global robust output reg-
We have applied a distributed internal model to convert the global output regulation problem into the global stabilization problem of an augmented system. Then we have further globally stabilized the augmented system via a distributed output feedback control law, thus leading to the solution of the global robust output regulation problem of the original system.

Our problem is a generalization of the leader-following problem in several ways. In particular, by introducing an exosystem, we allow the leader system to have different dynamics as the follower system. As a result, our control law can handle a class of reference inputs and a class of disturbances generated by the exosystem.

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