DIMENSIONAL AND NEIGHBOURHOOD DEPENDENCIES OF PHASE TRANSITIONS IN THE AXELROD CULTURE DISSEMINATION MODEL

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ABSTRACT

The spatial Axelrod model is a sociological system that can be studied quantitatively to investigate critical values of agent numbers and opinions to identify some of the complex interplay effects that arise when cultures meet and clash. We use an interdisciplinary mix of simulation programming techniques and complex systems measurements to investigate the effect of changing the spatial geometry, dimension and neighbourhood of locally interacting Axelrod agents that hold differing opinions or other sociological traits. We measure the change in the phase transition with the number of traits for two-, three- and for the first time, four-dimensional model systems. We find a nonlinear relationship with dimension and the critical number of traits. We find that although hexagonal and triangular lattices have different critical trait numbers from the conventional square lattice, using nearest or next-nearest neighbours or both, does not have a significant effect.

KEY WORDS
Culture dissemination; spatial model; spread; geometries.

1 Introduction

The Axelrod model of cultural dissemination [3] has been widely studied through numerical simulation in one [16] and two dimensions [13]. The model is based on the notion of homophily so that agents are posited to interact more frequently with those that share their properties and that social influence [27] drives a set of interacting agents to words uniformity or consensus. In his 1997 article [3] Axelrod noted some of the social mechanisms for driving the system to cultural similarity include: state formation; succession conflicts; transnational integration; domestic cleavages. Alternately, explanations for durable cultural differences in a system include: social differentiation; fads and fashions; preference for extreme views; drift; geographic isolation; specialization; changing environment or technology. Axelrod’s model is an agent-based spatial system with no central authority and it uses adaptive rather than rational agents [8].

Conventionally the Axelrod model has been studied on small systems [23] with square lattices of size typically ranging from $10 \times 10$ to $50 \times 50$ agents. Spatially, a random mixture of agents with differing opinions interact locally through a social peer-pressure based mechanism. The result is that spatial order emerges, with a rich set of complex spatial patterns of shared culture that under the right conditions, can lead to uniform consensus throughout the whole system.

There is a known phase transition in the spatial Axelrod model and if there are enough differently opinioned agents in the system then a trajectory towards a monoculture is impossible, with fractured barriers emerging spatially, that cannot be dispersed with the localised interaction rules. The location of this transition is well studied in two dimensions and some work has been reported on three dimensional Axelrod models [12]. No work has been reported in higher dimensions however, and the general dimensional dependence has not been known. In this present article we study the effect of the phase transition on higher dimensions including four, and also investigate the effect of using non-square lattices in two dimensions. We study hexagonal and triangular lattices and also consider how the two dimensional system changes behaviour when next-nearest and Moore neighbourhoods are employed instead of the conventional nearest-neighbour only.

The model has been studied in the limit of having only binary features [2] and this simplification allows for comparison with other well known thermodynamical models [9] exhibiting phase transitions such as the Ising model with Glauber dynamics [4]. Simpler models as the Ising model and the Sznajd opinion model have similar properties to Axelrod models and in one sense the Axelrod agent can be considered a superset model to some of the binary feature models.

Other reported work on the model has considered the effect of applying global media or social pressures that are not just localised [10, 24, 25]. Cultural drift influences [14] can also be incorporated as globally applied fields of influence and comparisons have been made between social media and applied model fields. Extensions to the fundamental nature of the Axelrod agent interactions has also been reported, with agents that accept localised change or discuss it first [7]. Cultural dynamics [31] can also be studied and attempts to incorporate Axelrod agent diffusion [26] and migration [29] effects have also been reported.

Most work on the Axelrod system has been on two dimensional systems although there has been revived recent interest in theoretical comparisons with other one-
dimensional models [17, 18] that allow investigation of the pathways to achieving uniform consensus. Some alternative geometries include generalised networks or graphs that are constructed statistically [5] or which have some geographic origin such as that from a network of European cities [6].

In summary, with interdisciplinary sociological modelling of continued interest, there has been considerable revived recent interest in the Axelrod spatial agent model. In this paper we try to understand the role and affect of geometry, dimension and neighbourhood on the phase transition in the number of traits in the classic Axelrod agent system. Our article is structured as follows: In Section 2 we define the Axelrod model and its parameters and describe our implementation of it for multi dimensional systems. We give selected screen shots characterising the model’s time evolution and various time and averaged properties computed from simulations in Section 3. In Section 4 we discuss the effect of the changed dimensionality and geometry on the phase transition, and offer some conclusions and ideas for further work in Section 5.

2 Simulation Method

Algorithm 1 Axelrod Cultural Dissemination Model

```java
public void evolveAxelrodModel(int steps ){
    if( uniform )return;

    int ender = step+steps;
    for(; step<ender; step++){
        uniform = true;

        for( Site a : sites )
            a.copy(); // copy to two-phase buffer

        for( Site a : sites )
            Site b = neighbours.random( a );

        for(int j=0; j<f; j++){
            if( a.features[j] != b.features[j] ){
                choices[nDiff++] = j;
            }
        }

        if( nDiff > 0 ){
            uniform = false;
            p = (double)(f-nDiff)/(double)f;
            if( rng.nextDouble() < p ){
                r = choices[rng.nextInt(nDiff)];
                a.buffer[r] = b.features[r];
            }
        }

        for( Site a : sites )
            a.update(); // copy from buffer

        if( uniform ) break;
    }
}
```

Figure 1. Java code to implement the multi-dimensional Axelrod model with high-level neighbour gathering utilities abstracting over the different geometries, lattices and dimensionalities of the system.

so that each and all of the features can each take on \( q \) values or traits. There is some disagreement in the research literature as to whether to use capitals or not for these parameters. In this present work we use \( F \) and \( f \), and also \( Q \) and \( q \) interchangeably.

The model is initialised randomly with each agent taking on any one of its \( q \) possible values with equal probability. It is evolved in discrete time-steps using Algorithm 1.

The cultural similarity probability \( P_{k,r} = n_{k,r}/f \) is chosen to model the idea that people or agents are more likely to interact with others who share many of their cultural attributes or beliefs and that the likelihood increases with the shared features. Like many spatial agent models, the Axelrod system is interesting because of the emergent bulk and large scale behaviour that emerges from ap-
application of relatively simple pairwise localised interaction rules.

There are a number of variations that can be made to the algorithm by using fixed or random order and in-situ or two-phase updating. To ensure there are no correlation artifacts introduced from sweeping effects we used a two phase update for the work reported here. We used an implementation of the Mersenne Twistor random number generator [20], which is of sufficient randomness quality that associated artifacts are unlikely to have been introduced into the trajectory samplings. We implemented the model using a customised multi-threaded Java program with built-in options to change the underpinning spatial lattice neighbourhood and geometry. Some measured properties are simple sums over the $N$ agents.

Figure 1 shows how the Axelrod model can be implemented in Java to abstract over the details of the lattice dimensionality, geometry and neighbourhood sizes. These are set up elsewhere in the simulation program as parameters of the run.

The time evolution of the model is important but the code to make measurements also needed to be computationally efficient. Component labelling to identify the size $S_{\text{max}}$ of the largest cluster of like-minded agents was implemented using a standard algorithm [11].

The code listed in Figure 2 illustrates how components can be first labelled with positive integers and how a map structure in Java that maps component label to component size can be used to build up a size distribution from with the maximum component size $S_{\text{max}}$ can be determined.

The system has $N$ agents in it. These can be arranged in a simple $50 \times 50$ square mesh, which is a size and arrangement commonly reported and discussed in the literature, or various other arrangements can also be investigated. We consider triangular and hexagonal lattices in two dimensions as well as hyper-cubic lattices in three and four dimensions. The model system with $f$ features each allowed to take on any of $q$ traits thus has $q^{fN}$ possible states overall. This is a large number and any simulation is only sampling trajectories through this space, which is infeasibly large to sample uniformly. Nevertheless it appears that numerical samplings do indicate convergent representative bulk behaviour as discussed below.

3 Computational Results

We present a selection of screen-dumps illustrating model behaviour and plots of various metrics as we investigate the various phase transitions in the Axelrod model when the number of traits is changed.

Figure 3 shows a time sequence of configurations of the Axelrod model simulated on a $200^2$ lattice. Only two features out of seven simulated are rendered - black/red indicates feature 1 when feature 2 is zero and green/blue indicates the two possible values of feature 1 when feature 2 is one. As can be seen, the initially random mixture forms connected component clusters of like minded agents as time progresses on a logarithmic scale. This particular combination of features (7) and traits (2) will form a single uniform component cluster after around 30,000 time steps. For a number of possible traits that is larger than the critical value $q^*$ however, the system has too many fractured boundaries of different feature/trait combinations and it becomes stuck in an unchanging or oscillating combination of many small component clusters.

The range of $q$ values in which this transition between occurrence of a large component and its absence for a given number of features, dimension, lattice geometry is surprisingly narrow. We use the long-term normalised size of the largest component cluster $S_{\text{max}}/N$ as a metric to study this phase transition for various conditions. It was first necessary to determine how long the simulations needed to be run to ensure that a correct measurement of $S_{\text{max}}/N$ was being made.

Figure 4 shows an averaged value of $S_{\text{max}}/N$ over ten independent randomly seeded initial configurations for a lattice size of $50^2$ and with $f = 10$ features. Ten features has become almost a canonical value that is widely used and reported in the literature on the Axelrod model. The figure shows the rapid rise in the fractional size of the largest component with time. Broadly speaking initial transient effects for different numbers of traits are all damped out after 30,000 - 40,000 time steps. The subsequent flat line regime justifies characterising the $f/q$ regime by a single averaged value of $S_{\text{max}}/N$. We run the experiments reported here for 100,000 steps or until complete agreement/uniformity is attained in the model system.

Similar temporal regimes are seen in a time step plot of the number of components $n_C$ present in the system. Figure 5 shows how this drops rapidly with time for various numbers of traits $q$.

It is also interesting to look explicitly at the time step value after which the system becomes uniform. Figure 6 shows this plotted against number of traits $q$ for the same...
double sMax = 0.0; // normalised size of largest cluster
...

public void computeComponents(){
    int i = 0;
    for( Site n : sites ) // initialise all with different component-labels:
        n.component = i++;

    boolean changed = true; // sweep the labels to their minimum allowable value:
    while( changed ){
        changed = false;
        for( Site n : sites ){
            for( Site m : neighbours.same(n) ){
                if( n.component != m.component ){
                    n.component = m.component = Math.min( n.component, m.component );
                    changed = true;
                }
            }
        }
    }

    nComponents = 0; // build up size histogram as a map:
    Map<Integer, Integer> sizeMap = new HashMap<>(); // maps component-label to component-size
    for( Site n : sites ){
        int c = n.component;
        if( sizeMap.containsKey(c) )sizeMap.put(c, sizeMap.get(c) + 1 );
        else{
            sizeMap.put(c, 1 );
            nComponents++;
        }
    }

    nIncMax = 0;
    for( int c : sizeMap.keySet() ) // for component labels actually present
        if( sizeMap.get(c) > nIncMax ){
            nIncMax = sizeMap.get(c);
            componentWithMax = c;
        }
    sMax = nIncMax / (double)N; // normalise to be a fraction of the system size
}

Figure 2. Java code demonstrating simple component labelling algorithm and size histogram identification of largest cluster size using a frequency map of sizes.

The s-curve in $n_C$ is asymmetric and therefore it is harder to assign a specific value to $q^*$. Figure 8 shows a plot of $S_{\text{max}}/N$ plotted against number of traits $q$ for the same system. The s-curve is symmetric and it is easier to assign the $q^*$ value from where $S_{\text{max}}/N$ crosses the middle 0.5 value. This procedure can then be adopted and applied to different values of $f$.

We investigate the effect of changing the number of different features $f$ to confirm that there are no anomalous transitions or other effects in the vicinity of the canonical value $f = 10$ reported in the literature.

Figure 9 shows the effect of some selected values of $F = 3, 5, 10, 15$ and as is seen, the s-curve in $S_{\text{max}}/N$ shifts right with increasing $F$ but does not change appreciably in shape. Large $F$ values do lead to larger experimental
Figure 3. Configurations of a $200^2$ model Axelrod system at time-steps 0; 100; 1,000; 10,000 - only 2 features out of 7 are shown, each has only $q = 2$ possible traits.

Figure 5. Average number of components plotted versus time-step, for various $q$.

Figure 6. Time step at which the system becomes uniform, plotted versus $q$.

Figure 7. Final number of components present in the system plotted versus $q$.

Figure 8. Size of the final largest component plotted versus $q$. 
errors - obtained from standard deviations computed over ten independent sample runs.

We show the results of experiments scanning across $F$ in Figure 10. There appears to be a linear relationship $q^* \approx 5.5 f$ for the two dimensional Axelrod model system over the range $f \approx 5 - 20$. This justifies focusing upon $F = 10$ as a parameter value to fix for investigation of other effects.

We can therefore investigate various lattice geometries, different neighbourhood combinations and different dimensionalities of the whole system.

Figure 11 shows the effect of applying different neighbourhoods and geometries in two dimensional systems. Surprisingly the model is insensitive to whether it uses nearest neighbour or next-nearest-neighbour or both (Moore neighbourhood) connectivities. Within the experimental errors attainable there is no discernible difference. However the model does show a marked difference when a triangular or hexagonal lattice is used instead of a square one. The Triangular lattice reduces the value of $q^*$ to around 45, whereas the hexagonal lattice geometry increases it to around 67 from the square lattice value of around 55.

These data were obtained from simulations on lattice sizes from $N = 50^2$ to $N = 200^2$ and are reasonably consistent and sharp with averages only over 10 sample runs.

4 Discussion

It is interesting to consider if there is a systemic relationship between the model dimensionality and $q^*$. For many models, there is a critical dimension itself, above which the model no longer has a phase transition or if it does, it can be explained simply using a mean-field theory based on local average spatial behaviours.

The work reported above on two dimensional systems was feasible using a few hundred processor hours of compute time. To investigate higher dimensions however requires considerably more processing time and we have only been able to investigate relatively small lattice lengths in higher dimensions. We investigate behaviour of nearest neighbour connectivity on Axelrod model systems on $20^3$ and $10^4$ lattice. Although Klemm and co-workers report work on three dimensional Axelrod systems, to our knowledge ours is the first reported work on a four dimensional system.

Figure 12 shows a comparison of 2d 3d and 4d model systems using the same procedure as described above. This suggests that the transitions are at approximately $q$ of 55, 133 and 222 respectively. This might indicate a simple linear relationship $q^* \approx 83.5 d - 114$ although this is very tentative, based on only three data points. There is clearly scope for work on a 5 or possibly 6 dimensional system.

The compute time to perform the simulation is largely
linear in the number of agents $N$. So far as we can tell there is no obviously worsening relationship between the necessary time-steps to attain uniformity or stickiness and dimension. We might expect some relationship with lattice length however as information about component membership must propagate across the system. This appears to be subsumed in the transient time regime.

We would also expect there to be finite size effects and that too small a lattice size makes it easier to attain consensus in the presence of many traits. One would expect our values of $q^*$ to be upper bounds and that in the thermodynamic limit of large system sizes, the values might be in fact smaller. There is scope for more extensive simulations on larger systems to verify a possible trend.

We use the size of the final largest component cluster of like-minded agents as a metric for studying the phase transition. The calculation of $S_{\text{MAX}}/N$ makes it a computationally feasible metric. Although component labelling is typically computationally expensive we do not need to perform the calculation very often - only at the end of each simulation run. It does not therefore dominate the processing time. Other metrics have been reported and would be worth investigating for measurements made more frequently such as during the transient regime or at regular time intervals.

The variation of the phase transition with dimension is possibly indicative that a localised theoretical model based on mean-field [30] values might be applicable to the Axelrod model. There does not appear to be a critical dimension at $d = 2$ that some models such as the Ising model exhibit. We have seen that the transitions in $q$ occur at quite high values however and it is therefore possible that some transition at higher dimensions occurs. The dependence on neighbourhood size is also non-trivial and we have seen from the presented results that the transition to a 6-neighbour hexagonal lattice is at a quite different location from that in the 6 nearest-neighbour cubic lattice system.

Other specialist geometries such as rewired lattices with small-world shortcuts are an interesting geometric variation for further experimentation [22]. The small-world phenomena is a well known sociological effect itself [21] and it is likely that introduction of spatially long-distance connections will have an effect on the location of the phase transitions. Small-world studies typically require quite large systems however [19]. They produce quite strong effects even from a very low re-wiring probabilities. To probe the requiring probability is likely to require several decades on lattice length scale [1]. This is not easy as shown from the large compute times required to reach the finalised uniform component cluster in the Axelrod model. Parallel computing and other high performance techniques will likely be required to enable a systemic small-world investigation of Axelrod model systems.

We have focused on simulations with fixed values of the number of features, following the conventional $f = 10$ value widely studied and reported in the literature. There is however scope for work to construct a more detailed phase diagram, verifying that there are no surprise transitions in $f$ itself at higher values.

5 Conclusion

We have studied the effect of changed dimensionality, changed agent interaction neighbourhood, and changed lattice geometry on the Axelrod spatial agent model of culture dissemination. We implemented a code that could change these properties and thus perform scan experiments in the number of traits $q$ to identify the location of the phase transition $q^*$. We found the nature of the local neighbourhood in two dimensions – nearest, next-nearest or both – did not affect the location of the transition significantly from a value of $q^*_{2d} \approx 55$.

However we found that the model is sensitive to geometric arrangement and that use of a triangular lattice (3 neighbours) reduced the location of the transition to $q^* \approx 45$ and use of a hexagonal lattice (6 nearest neighbours) increased the location of the transition to $q^* \approx 67$.

Changing the dimensionality of the system to 3 and hence also with 6 nearest neighbours has an even more marked effect on the phase transition, increasing it to $q^*_{3d} \approx 133$ and similarly in four dimensions increasing it to $q^*_{4d} \approx 222$. These values are likely upper bounds due to finite size effects that likely influence our necessarily small simulated lattice lengths. There appears to be a linear dependence of $q^*$ upon dimension $d$.

There is scope to experiment with larger model system sizes although this will require considerably more computational resource than the few hundred processor hours we have expended on this present study. Study of even small systems in higher dimensions such as five or six would be valuable to determine if the dimensional dependence is indeed linear.

A range of other realistic sociological effects such as committed individuals [28], immutable groups, and indi-
iduals with long memory effects [15] could also be incorporated into the Axelrod model. Other geometries such as small-world systems and semi-realistic geographic patterns arising from actual socio-political scenarios as state or international voting patterns are also worthy of study. We believe it is important and useful however to have clear and definite analyses of the regular lattice and multi-neighbour and dimensional geometries so that any such real world studies using the Axelrod model or one of its derivatives can be studied with a knowledge of what is a geometric effect and what is due to other introduced parameters.

References


