EVALUATION OF THE EFFECTS OF SLICING ANGLE ON VISCOELASTIC BIO-MATERIAL SLICING PROBLEM

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ABSTRACT
In this paper, we present the formulation of the stress distribution within a viscoelastic material subjected to distributed pressure on the free surface. In the formulation, the stress tensor at any point in the material is obtained with two steps. First, the elastic-viscoelastic correspondence principle is applied to obtain the stress solution for viscoelastic materials under a point load. Second, the obtained solutions from first step are integrated over the stress-applying area. Based on the formulation, the stress tensor at any point in the material is calculated numerically. This formulation can be used to simulate many practical contacting problems: one important example is the biomaterial cutting operations where a blade interacts with a biomaterial. In cutting, the effect of slicing angle on the stress distribution is an important factor to be included in the discussion. The slicing angle is determined by the magnitudes of tangential and normal components of the cutting force. Using the calculated principal stresses, it is possible to predict the location of damage using failure criterion such as Tresca’s criterion. The results can serve as guidelines for several applications where the stress distribution and fracture prediction in viscoelastic materials are concerned.

KEY WORDS
Linear viscoelastic, stress distribution, slicing angle, Tresca stress.

1. Introduction

Contact problems are frequently encountered in many situations. For example, in the design of a surgical simulator involving a scalpel, it is necessary to know the dynamic contacting force applied to the edge of the scalpel, in order to provide a vivid feeling of real surgeries. Such a force, once the stress distribution is known, can be calculated by integrating the stresses on the surface over the contacting area. In order to better understand a dynamic surgical process, it is also necessary to know the stress distribution in the material when the location of fracture initiation is a concern. In the process of biomaterial cutting, an important parameter is the slicing angle. With a proper selection of the slicing angle, the cutting process can be more efficient. In this paper, we use a linear viscoelastic isotropic half-space to represent the biomaterial body and model the cutting effect using distributive forces.

The initial study of contact problem could date back to 19th century when Boussinesq [1] solved the stress field of a half-space subjected to a vertical concentrated force in 1885, and Cerruti [2] organized the solution of half-space problem under a tangential concentrate force in 1882. The developments of these classical solutions were facilitated by practical needs. Selvadurai [3] extended Boussinesq’s classical solution to cover the case in which the half-space is reinforced with a fully bonded rigid inclusion of finite radius at a certain depth. Schepers, Savidis and Kausel [4] applied the classical solutions in a geotechnical background and considered the stress settlement within a soil caused by surface loading. Zhou and McMurray [5] used both classical solutions in modeling a biomaterial cutting problem in which the effects of blade were assumed to be the combination of a vertical distributive pressure and a tangential distributive stress.

Another extension is to consider the problem for viscoelastic medium. Talybly [6] solved the viscoelastic Boussinesq’s problem in which the medium is linear viscoelastic. Peng and Zhou [6] solved the viscoelastic Cerruti’s problem and generalized the case in which combined loading conditions are applied. Adolph et al. [8] numerically evaluated the stress and displacement fields for a 3D viscoelastic half-space subjected to a rectangular surface vertical loading using Radon and Fourier Transforms.

Soft biological tissues are typical viscoelastic materials, and the mechanical loading capability of soft biological tissues is an important determinant of degeneracy or injuries. Studies in viscoelastic failure or fracture problems were also investigated by many researchers [9][10][11][12]. The authors of this paper are interested in obtaining the viscoelastic solutions and applying them to clinical applications, in which the deformation, stress and failure of soft biological tissues are studied. By studying the response of biomaterial under various impact conditions, this paper aims to link the clinical observations to biomechanical modeling, provide explanations to clinical operations and serve as guidelines for biomedical researches.

With the help of Matlab, we calculate the stress field numerically as well as Tresca stress (maximum shear stress). The Tresca stress fields are plotted under different slicing angles (0°, 10°, 30°, 60° and 85°) in two planes: \( O-xz \), which is along the blade, and \( O-yz \), which is...
perpendicular to the blade. The magnitude of Tresca stress determines the cutting efficiency.

2. Modeling and Solution Derivation

2.1 Model Description

The blade-biomaterial contact problem is simplified as the model shown in Figure 1, where the tangential and normal force components are applied over a rectangular area on the surface of the half-plane. The o-xyz is a frame fixed on the half-plane; A(x, y, z) is a point in the material with coordinate x, y, z; A'(x, y, 0) and B(ζ, η, 0) are two points on the surface; the distance from B to A' is r and from B to A is R; The normal and tangential components of the applied distribution force are \( p_n \) and \( p_t \), respectively; and the angle between \( p_n \) and the resultant pressure is \( \alpha \), which is defined as slicing angle in this paper.

The surface integration is performed using the scheme in Figure 1 over the rectangular region \( S \) (-a<ζ<a, -b<η<b), which represents the contacting area. The results are given in the form of stress tensor: \( \sigma \) = \([\sigma_{xx}, \sigma_{xy}, \sigma_{xz}; \sigma_{yx}, \sigma_{yy}, \sigma_{yz}; \sigma_{zx}, \sigma_{zy}, \sigma_{zz}]\).

![Figure 1. Simplified model](image)

2.2 Stress Distributions

The classic problem of an elastic half-space subjected to a point load was solved by Boussinesq [1] (Boussinesq’s problem, in which a vertical point load is applied) and Cerruti [2] (Cerruti’s problem, in which a tangential load is applied). Under linear viscoelastic assumption, we directly derived the solutions for viscoelastic half-space subjected to point tangential and normal load by applying the elastic-viscoelastic correspondence principle. To obtain the solutions under uniform distributed pressure and tangential force, the surface integration is performed using the scheme in Figure 1 over the rectangular region \( S \).

Under the linear viscoelasticity assumption, the stress field can be calculated as the superimposition of two independent stress fields: one yielded by tangential distributive forces \( P_t(x,y,t) \) and another one by vertical distributive forces \( P_n(x,y,t) \). The magnitude of these two forces are determined by the magnitude of the actual external force \( P(x,y,t) \) and the slicing angle \( \alpha \). Their relationships are written in (1).

\[
\begin{align*}
P_t(x,y,t) &= P(x,y,t) \cos \alpha \\
P_n(x,y,t) &= P(x,y,t) \sin \alpha
\end{align*}
\]

The stress field due to vertical distributive forces, denoted by superscript \( n \) is shown in (2), and the stress field due to tangential distributive forces, denoted by superscript \( t \) is shown in (3). Noted is that, all stress components and external forces are functions of spatial variables \( x, y \) and \( z \) and time \( t \). A material function \( V \) appears in the solution representing the viscoelasticity, and it is a function of time \( t \). The derivation of \( V(t) \) is given in section 2.3.

\[
\begin{align*}
\sigma_{xx} &= \frac{3x^2z}{2\pi R^3} P_t - \frac{1}{2\pi} \left[ \frac{x^2-y^2}{R^3(R+z)} + \frac{y^2z}{R^3} \right] P_n V \\
\sigma_{yy} &= \frac{3y^2z}{2\pi R^3} P_t - \frac{1}{2\pi} \left[ \frac{y^2-x^2}{R^3(R+z)} + \frac{x^2z}{R^3} \right] P_n V \\
\sigma_{zz} &= \frac{3z^3}{2\pi R^3} P_t \\
\sigma_{xy} &= \frac{3xyz}{2\pi R^3} P_t - \frac{1}{2\pi} \left[ \frac{xy}{R^3} \left( \frac{xy}{R^2} \right) \right] P_n V \\
\sigma_{xz} &= \frac{3xyz}{2\pi R^3} P_t - \frac{1}{2\pi} \left[ \frac{xy}{R^3} \left( \frac{xyz}{R^2} \right) \right] P_n V \\
\sigma_{yx} &= \frac{3xyz}{2\pi R^3} P_t - \frac{1}{2\pi} \left[ \frac{xy}{R^3} \left( \frac{xyz}{R^2} \right) \right] P_n V \\
\sigma_{zx} &= \frac{3xyz}{2\pi R^3} P_t - \frac{1}{2\pi} \left[ \frac{xy}{R^3} \left( \frac{xyz}{R^2} \right) \right] P_n V
\end{align*}
\]

where

\[
P(t)V = P(t)V(t) = \int_0^t P(t)V(t-\tau)d\tau
\]

2.3 Relaxation Modulus

The function \( V(t) \) represents the viscoelasticity of the material, and can be determined by the elastic-viscoelastic correspondence principle. The counterpart of \( V \) in the elastic solution is \( 1-2\nu \), where \( \nu \) is the Poisson’s ratio. From the relationship between elastic material constants,
we can express 1-2\nu in (4), where \(G\) and \(K\) denote the shear modulus and bulk modulus.

\[
1 - 2\nu = \frac{3G}{G + 3K}
\]  
(4)

Due to the viscoelasticity, 1-2\nu should be replaced with a time-dependent relaxation modulus, which is defined as \(V(t)\). As we already have the shear and bulk relaxation moduli, which are modeled using exponential decay functions, we can obtain the role of 1-2\nu in viscoelastic case from the RHS of equation (4). According to the elastic-viscoelastic correspondence principle [13], Laplace transform should be applied to (4), and the result is shown in (5).

\[
\hat{V}(s) = \frac{3\hat{G}(s)}{\hat{G}(s) + 3\hat{K}(s)}
\]  
(5)

Taking inverse Laplace transform to (5), the time-domain expression of \(V(t)\) can be obtained, and will appear in the convolution terms in solution (2) and (3).

3. Results

With the help of Matlab, we were able to numerically calculate the dynamic stress tensor for any given point \((x,y,z)\) in the half-space. We selected the following slicing angles: 0° (vertical), 10°, 30°, 60° and 85°. To evaluate the effects of slicing angle on the cutting process, we introduced the Tresca’s Criterion as a mean to determine where the initiation of fracture is potential. The Tresca stress was calculated as \(\tau = 0.5(\sigma_3 - \sigma_1)\), where \(\sigma_3\) and \(\sigma_1\) are the largest and least principal stresses. The critical shear stress \(\tau_C\) of the biomaterial can be experimentally determined, and the prediction of failure can be made by simply comparing \(\tau\) and \(\tau_C\). We are interested in stress distributions in the plane along the blade cutting direction: \(O-xz\), and the plane perpendicular to the blade: \(O-yz\). The plots of Tresca stress field in these two planes are shown in Figure 2-11. Noted is that, the stress has the same unit as the external force. In our paper, the units are KPa.

In the plane along the blade, the maximum Tresca stress occurs at the two ends of the blade \((x = a\) and \(x = -a)\). The magnitude increases when the slicing angle increases, meaning that the more tangential forces applied, the higher efficiency for the cutting process. In Figure 2, the maximum Tresca stress under vertical loading is only about 0.04KPa, which is much lower compared with Figure 6, where the slicing angle is 85° and the maximum Tresca stress is about 0.11KPa.
In the plane perpendicular to the blade, the maximum Tresca stress occurs when the slicing angle is 30° at about the depth of 0.1\(b\), as shown in Figure 9. The magnitude shows a different pattern from in \(O\text{-}xz\) plane. Tresca stress reaches a peak value when the slicing angle is 30° and then drops when the slicing angle keeps increasing. Noted is that the maximum Tresca stress in \(O\text{-}yz\) plane is much lower than in \(O\text{-}xz\) plane, meaning the damage will not initiate in plane with constant \(x=0\), but in the two ends of the blade. Also, the maximum Tresca stress does not occur in the surface, but in a certain depth, as shown in Figure 9, meaning the damage will not directly initiate in the contacting surface. This observation shows the same pattern as in elastic bodies.
4. Conclusion

Based on these formulations and data, the stress for any given point inside the half-space can be both analytically and numerically calculated for any plane which we are interested in, such as plane $O-xz$, which is parallel to the direction of tangential forces, or plane $O-yz$, which is perpendicular to the tangential forces. Knowing the stress tensor $[\sigma]_{3x3}$, we are able to obtain the principal stress and predict the failure using Tresca’s criterion, which requires the calculation of maximum shear stress. Previous research [14] has concluded that the maximum shear stress was the dominant reason in material failure, and we have successfully verified this conclusion for viscoelastic medium.

The dynamic stress distribution can be used to design a surgical simulator with a scalpel involved. The solution we developed can help trainees view the stress map due to each movement during an operation. It is also feasible to predict the location of potential damage if the failure data is available.

References