A STATISTICAL INFORMATION METHOD FOR SENSOR-TARGET GEOMETRY

Sung-Ho Kim & Joon Ha Park
Department of Mathematical Sciences
Korea Advanced Institute of Science and Technology
Daejeon, S. Korea
email: sung-ho.kim@kaist.edu & pjhtoto@hotmail.com

ABSTRACT
In this paper, the target localization problem using multiple sensors based on the time difference of arrival (TDOA) data is investigated under the assumption that the target is far off from the sensors. We examine the geometric features of the problem and compute the Fisher information matrix (FIM) and the Cramér-Rao lower bounds by using the power series expansion to analyze the variability of the angle and the range estimates. We found a relationship between sensor formation and tracking performance through the formula of the FIM.

KEY WORDS
Time difference of arrival, log likelihood function, Fisher information, Cramer-Rao lower bound, linear approximation, Gaussian noise.

1 Introduction and background

Target localization is used in a wide array of problems of various scales. A few examples include the global positioning system (GPS) using satellite communication, the passive tracking of a signal-emitting military target, and the locating technology of emergency callers. Various localization methods of different physical natures have been developed and studied[1]. The time difference of arrival (TDOA) method uses the measured time differences of the signals received by a set of sensors, and the frequency difference of arrival (FDOA) method utilizes the frequency Doppler shift phenomenon which arises when there exists a relative movement between the source and the receiver. The angle of arrival (AOA) method makes use of the difference in the direction to the target from the sensors at different locations.

Analysis of the localization problem based on different types of measurement is abound in literature, and algebraic equations characterizing the problem have been derived. Several methods for finding their solutions were suggested and their stochastic properties were discussed. The total least square (TLS) estimation[2] and the Taylor series expansion[3] are among the most commonly used techniques for TDOA localization.

Difficulty in localization is due to the nonlinearity of the estimating equations. [1] formulated the estimation problem as a least squares problem with constraints by transforming the quadratic distance relations into linear matrix identities with redundant parameters. The performance of this method was discussed by analyzing the approximate variances of the estimates obtained by applying the method used for unconstrained least squares estimators.

The studies on the localization problem thus far have focused on formulating estimating equations, which are functions of the target location and the measurement vectors, and on suggesting ways of finding their algebraic solutions. However, these approaches have barely addressed the geometric nature of the localization problem and the dependence of tracking performance on sensor formation. Moreover, in the cases where the target is located far off from the sensors, the estimating equations tend to become ill-posed and the solutions become increasingly sensitive to the observations. Thus, a new way of understanding the equations is needed for further practical analysis.

In this paper, we analyze the target localization problem by decomposing it into the range and the angle components and calculate the estimating performance of each component under the assumption that the target is located far off compared with the between-sensor distances. Our analysis is closely linked to the intrinsic features of the sensor-target geometry and reveals the relationship between tracking performance and sensor formation. In order to provide with mathematical basis, we first stochastically model the tracking problem in a general setting, and derive the Cramér-Rao lower bound (CRLB) for the variances of the range and the angle estimates based on this model under a Gaussian noise assumption. For long-range targets, the geometric meaning of the CRLB becomes clear once we express it in power series expansion. Finally, based upon this approximation, we develop a method for numerically estimating the target location. We present the details of this analysis for TDOA type measurement data assuming that all the sensors and the target are on the same plane, and we extend the result in a brief manner to the 3 dimensional space.

2 Preliminaries

Consider there are \((n + 1)\) sensors, numbered 0 through \(n\). Denote the distance from the \(i\)-th sensor to the \(j\)-th sensor by \(d_{i,j}\), and the distance to the target by \(d_{i,0}\). These \((n +
1) sensors are used for tracking the target location by the TDOA method. Denote the range (distance to the target) difference between 0-th and \( i \)-th sensors by \( r_i = d_{i,0} - d_{i,t} \).

![Diagram showing range difference](image)

**Figure 1.** Estimated range difference with the reference sensor at the origin

Now assume that the distance to the target is much greater than the distances between the sensors, as is often the case when multiple guided missiles are tracking a target. For convenience’ sake, we will call this assumption long-range assumption. Let the direction from the 0-th sensor to the target with respect to a fixed reference direction be represented by the angle \( \theta_i \), and the direction from the 0-th sensor to the \( i \)-th sensor with respect to the same reference direction by \( \theta_i \). Then, under the long-range assumption, the range difference \( r_i \) is approximated as (see Figure 1)

\[
r_i \approx d_{i,0} \cos(\theta_i - \theta_t). \tag{2.1}
\]

Thus, assuming that we know the location of all the sensors, (2.1) enables us to compute the target direction \( \theta_i \), once we calculate the range difference \( r_i \) by multiplying the measured time difference to the speed of light. This implies that each pair of sensors work as an angle detector under the long-range assumption.

### 3 Probability model

Let \( z_i, i = 0, \cdots, n \) denote the product of the speed of light and the time point at which the sensor \( i \) received a signal emitted from the target. This \( z_i, i = 0, \cdots, n \) be the differences in the range measurements with respect to sensor 0. Assuming that the only source of error is the measurement error and that the measurement made by a sensor is independent of each other, we can write the joint probability density function of the measurements, \( z_0, \cdots, z_n \), as

\[
h(z_0, \cdots, z_n | t, c) = f_0(z_0 - c - d_{0,t}) \cdots f_n(z_n - c - d_{n,t})
\]

where the location of the target \( t \) and the time shift, \( c \), are considered as parameters and \( f_i \) is the probability density function (or pdf) for the measurement error of the \( i \)-th sensor.

Under this model, the vector of measured range differences \( y = (y_1, \cdots, y_n) \) is an ancillary statistic for the time shift parameter, \( c \). To see this, note that \( y \) follows the distribution with its pdf

\[
f_{Y_1,\cdots,Y_n}(y_1, \cdots, y_n) = \int_{-\infty}^{\infty} f_0(z_0 - c - z_0) f_1(z_0 - y_1 - c - d_{1,t}) \cdots f_n(z_0 - y_n - c - d_{n,t}) dz_0,
\]

since the Jacobian \( \frac{\partial(z_0, z_0, \cdots, z_n)}{\partial(y_1, y_2, \cdots, y_n)} \) = 1. Then for any other time shift \( c' \), the change of variable \( z_0' = z_0 - c' + c \) yields

\[
\int_{-\infty}^{\infty} f_0(z_0 - c' - d_{0,t}) f_1(z_0 - y_1 - c' - d_{1,t}) \cdots f_n(z_0' - y_n - c - d_{n,t}) dz_0'.
\]

Therefore, we can see that the joint distribution of \( y \) is independent of the time shift parameter \( c \).

For simplicity, let us assume from now on that the measurement error of the sensors follows a normal distribution with covariance matrix \( \Sigma = \text{diag}(\sigma_0^2, \cdots, \sigma_n^2) \). The joint pdf equals

\[
h(z_0, \cdots, z_n | t, c) = \frac{1}{\sqrt{2\pi}^{n+1} |\Sigma|^{1/2}} \exp\left(-\frac{(z_0 - c - d_{0,t})^2 + \cdots + (z_n - c - d_{n,t})^2}{2\sigma_n^2}\right).
\]

Let \( A \) be the \( n \times (n + 1) \) matrix

\[
\begin{pmatrix}
1 & -1 & 0 & \cdots & 0 \\
1 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 0 & \cdots & \cdots & -1
\end{pmatrix},
\tag{3.1}
\]

then the range difference vector \( y \) can be written as

\[
y = \begin{pmatrix} z_0 - z_1 \\ \vdots \\ z_0 - z_n \end{pmatrix} = Az.
\tag{3.2}
\]

Thus the distribution of \( y \) is given by

\[
y \sim N(\text{r}(t), A\Sigma A') \tag{3.3}
\]

where \( r(t) = (r_1(t), \cdots, r_n(t)) = A \cdot (c + d_{0,t}, \cdots, c + d_{n,t})' = (d_{0,t} - d_{1,t}, \cdots, d_{n,t} - d_{n,t})' \). Note that this distribution does not depend on the time shift \( c \) as anticipated. Consequently, given the observed values \( y_i, i = 1, \cdots, n \), minus twice the log likelihood equals

\[
\lambda(t; y) = -2\log L(t|y) = \beta + (y - \text{r}(t))' (A\Sigma A')^{-1} (y - \text{r}(t)),
\tag{3.4}
\]

\text{(3.1)}
for some constant β. Let \( \Gamma = (A\Sigma A')^{-1} \). If all of the sensors have the same measurement errors, that is, if the variances \( \sigma_i^2 \) are all the same,

\[
\Gamma = \sigma_0^{-2}(I + \mathbb{1})^{-1} = \sigma_0^{-2}(I - \frac{1}{n+1} \mathbb{1}) \quad (3.5)
\]

where \( \mathbb{1} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}_{n \times n} \), so we have

\[
\lambda(t, y) = \sigma_0^2 \beta + \sum_{i=1}^{n} (y_i - r_i(t))^2 - \frac{1}{n+1} \left( \sum_{i=1}^{n} (y_i - r_i(t))^2 \right). \quad (3.6)
\]

Thus, the maximum likelihood estimator (MLE) of the target location \( t \) can be computed by solving the estimating equation obtained by differentiating Equation (3.4), which yields the estimation equations,

\[
(y - r(\hat{t}))' \Lambda \Delta t r(\hat{t}) = 0, \quad (3.7)
\]

where \( \Delta t = \left( \frac{\partial}{\partial d}, \frac{\partial}{\partial \theta} \right)' \) is the differential operator.

Writing the location of the target in polar coordinates, \( t = (d_t, \theta_t) \), and the derivative with respect to \( t \) accordingly as

\[
\Delta t r(\hat{t}) = \left( \frac{\partial r}{\partial d_t} (\hat{t}), \frac{\partial r}{\partial \theta_t} (\hat{t}) \right),
\]

we have a pair of unknown parameters and a pair of equations. Thus one can at least numerically solve these equations.

4 Estimation and sensor formation

4.1 Cramer-Rao lower bound and Fisher Information

In this section, we are concerned about the variability of the estimators of the target location. Suppose we are interested in finding a value of the parameter \( t \) that minimizes \( \lambda(t, y) \). Then the estimate \( \hat{t} \) is determined by solving the estimating equation (3.7). The sensitivity of the estimator \( \hat{t} \) with respect to the observation \( y \) is indicated by the value

\[
\left. \frac{dt}{dy} \right|_{t=\hat{t}} = \left. \frac{\partial \lambda}{\partial y} \right|_{t=\hat{t}} \left/ \left. \frac{\partial \lambda}{\partial t} \right|_{t=\hat{t}} \right.
\]

Thus, a large value of \( \left| \frac{\partial \lambda}{\partial y} \right| \) would in general imply that the estimator is less sensitive to the observation and hence has smaller variability. The CRLB accounts for this point in a more rigorous fashion. The variance of any statistic \( W \) where the distribution is parameterized by \( \eta \) is bounded from below by the CRLB under some conditions on the distribution[4]:

\[
\text{Var}(W) \geq \left( \frac{d}{d\eta} E_\eta W \right) I(\eta) \left( \frac{d}{d\eta} E_\eta W \right)',
\]

where the inequality \( A \geq B \) is to be understood as the matrix \( (A - B) \) is positive definite. The term \( I(\eta) := -E_\eta D_\eta^2 \log f(x; \eta) \) is called the Fisher information matrix (FIM) of the distribution. It should be noted that maximum likelihood estimators satisfy consistency and asymptotic efficiency under reasonable regularity conditions[5].

From (3.3), it follows that the Fisher information matrix equals

\[
I(t) = E_t (\Delta t \log f(y; t)') (\Delta t \log f(y; t)) = E_t (\langle y - r(t) \rangle \Lambda (\Delta t r) (\langle y - r(t) \rangle \Lambda (\Delta t r)) = (\Delta t r)' \Lambda (\Delta t r).
\]

(4.1)

A power series expansion of the range difference \( r_i \) in terms of \( \frac{d}{d\tau} \) makes a simple but useful approximation of the Fisher information matrix. We think of the target and the sensors on the complex plane whose coordinates are denoted by \( \tau \) and \( \omega_i, i = 0, 1, \cdots, n \). Then if the target is at \( (x_t, y_t) \) in Cartesian coordinates, \( \tau = x_t + jy_t \) where \( j = \sqrt{-1} \). Let the distance from the origin to the target and to the \( i \)-th sensor be denoted by \( d_i \) and \( d_{i,t} \). Then, since \( |\tau| = d_t, |\omega_i| = d_i \), and \( |\tau - \omega_i| = d_{i,t} \), we have

\[
r_i := d_{0,t} - d_{i,t} = d_i (|\tau - \omega_0| - |\tau - \omega_i|).
\]

Note that \( r_i \) is a simplified notation of \( r_i(t) \) in (3.6)

Thus,

\[
\left. \frac{r_i}{\tau} \right|_{\tau=\hat{\tau}} = 1 - \frac{d_i}{d_t} \cos(\theta_t - \theta_t) + \frac{1}{2} \frac{d_i^2}{d_t^2} \sin^2(\theta_t - \theta_t)
\]

\[
+ \frac{1}{2} \frac{d_i^2}{d_t^2} \cos(\theta_t - \theta_t) \sin^2(\theta_t - \theta_t) + O(\frac{d_i^3}{d_t^3}),
\]

we have

\[
\frac{r_i}{\tau} = d_i \cos(\theta_t - \theta_t) - d_0 \cos(\theta_0 - \theta_t)
\]

\[
- \frac{1}{2} \frac{d_i^2}{d_t^2} (d_i^2 \sin^2(\theta_t - \theta_t) - d_0^2 \sin^2(\theta_0 - \theta_t))
\]

\[
- \frac{1}{2} \frac{d_i^2}{d_t^2} (d_i^2 \cos(\theta_t - \theta_t) \sin^2(\theta_t - \theta_t))
\]

\[
+ d_0 \cos(\theta_0 - \theta_t) \sin(\theta_0 - \theta_t) + d_i \cdot O(\frac{d_i^3}{d_t^3}).
\]

(4.2)
Differentiating equation (4.2), we obtain
\[
\frac{\partial r_i}{\partial \theta} = d_i \sin(\theta_i - \theta) - d_0 \sin(\theta_0 - \theta) + \frac{1}{2d_t} (d_i^2 \sin(2\theta_i - 2\theta) - 2d_0 \sin(2\theta_0 - 2\theta)) + d_i \cdot O\left(\frac{d_i^2}{d_t^2}\right),
\]
\[
\frac{\partial r_i}{\partial d_t} = \frac{1}{2d_t^2} (d_i^2 \sin^2(\theta_i - \theta) - d_0^2 \sin^2(\theta_0 - \theta)) + \frac{1}{d_t} (d_i \cos(\theta_i - \theta_0) \sin^2(\theta_i - \theta_i))
\]
\[
- d_0^2 \cos(\theta_0 - \theta_0) \sin^2(\theta_0 - \theta_0)) + O\left(\frac{d_i^2}{d_t^2}\right)
\]

If we plug in these approximations into (4.1), the Fisher information matrix becomes
\[
I(t) = \begin{bmatrix}
I_{dd} & I_{d\theta} \\
I_{d\theta} & I_{\theta\theta}
\end{bmatrix}
\]
with
\[
\begin{align*}
I_{dd} &= \sigma_0^{-2}(M_4 - \frac{1}{n+1} N_2^2) / 4 d_t^4, \\
I_{d\theta} &= \sigma_0^{-2}(M_3 - \frac{1}{n+1} N_1 N_2) / 2 d_t^2, \text{ and} \\
I_{\theta\theta} &= \sigma_0^{-2} M_2 - \frac{1}{n+1} N_1^2
\end{align*}
\]
where
\[
\begin{align*}
M_4 &= \sum_{i=1}^n (d_i^2 \sin^2(\theta_i - \theta) - d_0^2 \sin^2(\theta_0 - \theta))^2, \\
M_3 &= \sum_{i=1}^n ([d_i^2 \sin^2(\theta_i - \theta) - d_0^2 \sin^2(\theta_0 - \theta)]) \\
&\times (d_i \sin(\theta_i - \theta) - d_0 \sin(\theta_0 - \theta))] , \\
M_2 &= \sum_{i=1}^n (d_i^2 \sin(\theta_i - \theta) - d_0^2 \sin(\theta_0 - \theta))^2, \\
N_2 &= \sum_{i=1}^n d_i^2 \sin^2(\theta_i - \theta) - d_0^2 \sin^2(\theta_0 - \theta), \\
N_1 &= \sum_{i=1}^n d_i \sin(\theta_i - \theta) - d_0 \sin(\theta_0 - \theta).
\end{align*}
\]

(4.5)

under the assumption that the measurement errors are i.i.d. normal, i.e., $\Sigma = \sigma_0^2 I$.

When the reference sensor is located at the origin, i.e., $d_0 = 0$, (4.5) simplifies to
\[
\begin{align*}
M_4 &= \sum_{i=1}^n d_i^4 \sin^2(\theta_i - \theta), \\
M_3 &= \sum_{i=1}^n d_i^3 \sin^3(\theta_i - \theta), \\
M_2 &= N_2 = \sum_{i=1}^n d_i^2 \sin^2(\theta_i - \theta), \\
N_1 &= \sum_{i=1}^n d_i \sin(\theta_i - \theta). \\
\end{align*}
\]

(4.6)

Hence, the standard deviation of the range estimator is inversely proportional to the spread of the square of $d_i \sin(\theta_i - \theta_t)/d_t$, and that of the angle estimator is inversely proportional to the spread of $d_i \sin(\theta_i - \theta_0)$ when $d_0 = 0$.

4.2 Sensor formation efficiency

The results of Section 4.1 provide useful information about the efficiency of sensor formations. A formation of sensors can be regarded as more efficient than another if the estimated target location has smaller variability.

Equation (4.5) says that $M_2$, $M_3$, $M_4$, $N_1$, and $N_2$ are functions of only $d_i \sin(\theta_i - \theta_t)$, for $i = 0, \ldots, n$. This shows that the variance of the angle estimate is only dependent on the spread of the sensors along the line perpendicular to the target direction (see Figure 2). Also, note that when the reference sensor is at the origin and the formation of the sensors is symmetric with respect to the target direction, $N_1 = M_3 = 0$. In this case, we can see in (4.3) that $I_{d\theta} = 0$ making the Fisher information matrix diagonal. To investigate the relationship between estimation error and sensor formation we compared the sample variance of the estimated target positions with the CRLBs. Figure 3 shows the six sensor formations we used for the simulation experiment. The target was fixed at $(40, 40)$, and 200 measurements were randomly generated for each formation. The CRLBs were computed as the inverse of the Fisher information matrix (4.3).

Figure 4 shows an experiment result with $\sigma_0 = 0.001$. The six points are mostly on the $y = x$ line indicating that sample variances are almost equal to the CRLBs, which were simply calculated based on the sensor formations. Therefore, given the target location, the sensor formation, and the size of the measurement error, we can approximately predict the performance of the sensor formation.

The between-sensor distances in formations 1 and 2 are half of those in formations 5 and 6. This difference is reflected in the variances of the estimates in Figure 4 for both of the angle and the range estimates. Formations 3 and 4 are with more sensors than formation 1 and their estimate variances are smaller than those of formation 1.
4.3 Fisher information matrix in 3-dimensional target tracking

In case that the target and sensors are at different altitudes, we need to extend our discussion to the 3-dimensional space. We use a power series approximation of $r$ in the spherical coordinate system, as in Section 4.1. Let $i = (d_i, \phi_i, \theta_i)$ be the coordinate vector of the location of the target, where $\phi_i$ is the inclination angle and $\theta_i$ is the azimuth angle, and $\mathbf{i} = (d_i, \phi_i, \theta_i)$ the location of $i$-th sensor. For simplicity, assume that the 0-th sensor is located at the origin. Then, the range difference is exactly

$$r_i := d_t - d_{t,i} = d_t \left(1 - \sqrt{Q}\right)$$

where

$$Q = 1 - 2 \sin \phi_i \sin \phi_t \frac{d_t}{d_i} \cos(\theta_t - \theta_i) - 2 \frac{d_t}{d_i} \cos \phi_t \cos \phi_i + \frac{d_i^2}{d_t^2}.$$ 

The power series in terms of $\frac{d_i}{d_t}$ yields

$$r_i = \frac{d_i}{d_t} \left\{ \sin \phi_t \sin \phi_i \cos(\theta_t - \theta_i) + \cos \phi_t \cos \phi_i \right\} - \frac{d_t^2}{2d_i} \left\{1 - \sin^2 \phi_t \sin^2 \phi_i \cos^2(\theta_t - \theta_i) \right. \\
- \cos^2(\phi_t) \cos^2(\phi_i) - 2 \sin \phi_t \cos \phi_i \sin \phi_i \\
\times \cos \phi_t \cos(\theta_t - \theta_i) \right\} + d_t O\left(\frac{d_i^3}{d_t^3}\right).$$

Hence,

$$\frac{\partial r_i}{\partial d_t} = \frac{d_i^2}{2d_t^2} \left\{1 - (\sin \phi_t \sin \phi_i \cos(\theta_t - \theta_i) + \cos \phi_t \cos \phi_i) \right. \\
\times \cos \phi_t)^2 \right\} + O\left(\frac{d_i^3}{d_t^3}\right),$$

$$\frac{\partial r_i}{\partial \phi_t} = d_i (\cos \phi_t \sin \phi_i \cos(\theta_t - \theta_i) - \sin \theta_i \cos \phi_i) + d_i O\left(\frac{d_i}{d_t}\right),$$

$$\frac{\partial r_i}{\partial \theta_t} = d_i \sin \phi_t \sin \phi_i \sin(\theta_t - \theta_i) + d_i O\left(\frac{d_i}{d_t}\right).$$

Since

$$I(t) = \begin{pmatrix} I_{dd} & I_{d\phi} & I_{d\theta} \\
I_{\phi d} & I_{\phi\phi} & I_{\phi\theta} \\
I_{\theta d} & I_{\theta\phi} & I_{\theta\theta} \end{pmatrix} = (\Delta_t \mathbf{r}) \Lambda (\Delta_t \mathbf{r})'$$

where $\Delta_t = \left(\frac{\partial}{\partial \phi_t}, \frac{\partial}{\partial \theta_t}, \frac{\partial}{\partial \phi_i}\right)'$, we can derive that

$$I_{dd} = \frac{1}{4d_t^4} \left[ \sum_{i=1}^{n} d_i^2 \left\{1 - (\sin \phi_t \sin \phi_i \cos(\theta_t - \theta_i) \right. \\
+ \cos \phi_t \cos \phi_i)^2 \right\} - \frac{1}{n+1} \left(\sum_{i=1}^{n} d_i^2 \left\{1 - \right. \right. \\
\times \left. \left. \sin \phi_t \sin \phi_i \cos(\theta_t - \theta_i) + \cos \phi_t \cos \phi_i)^2 \right\} \right]^2, \right.$$
\[ I_{\phi\phi} &= \sum_{i=1}^{n} d_i^2 (\sin \phi_i \cos (\theta_i - \theta_i) - \tan \phi_i \cos \phi_i)^2 \\
&\quad - \frac{1}{n+1} \left( \sum_{i=1}^{n} d_i (\sin \phi_i \cos (\theta_i - \theta_i) - \tan \phi_i \cos \phi_i) \right)^2 \cdot \cos^2 \phi_i, \quad \text{and} \\
I_{\theta\theta} &= \sum_{i=1}^{n} \left( \sum_{i=1}^{n} d_i^2 \sin^2 \phi_i \sin^2 (\theta_i - \theta_i) - \frac{1}{n+1} \right) \left( \sum_{i=1}^{n} d_i \sin \phi_i \sin (\theta_i - \theta_i) \right)^2 \cdot \sin^2 \phi_i. \]

To distinguish the above \( I \) values from those in (4.3), we will denote the above \( I \) by \( I^3 \) to symbolize the \( I \) in the 3 dimensional space. We can easily check that, when we put all the values of \( \phi \) equal to \( \pi/2 \), i.e., all the sensors and the target are on the \( xy \)-plane, \( I_{\phi\phi}^3 \) and \( I_{\theta\theta}^3 \) values are exactly the same as the \( I \) values in (4.3).

We will call the line connecting the reference sensor and the target the \textit{target line}, and call the projection of the target line onto the \( xy \)-plane the \textit{xy-target line}. \( |d_i \sin \phi_i \sin (\theta_i - \theta_i)| \) is then interpreted as the projection of the distance of sensor \( i \) to the \( xy \)-target line onto the \( xy \) plane. \( I_{\phi\phi}^3 \) is affected by the spread of this values. As for \( I_{\theta\theta}^3 \), we look into \( d_i (\sin \phi_i \cos (\theta_i - \theta_i) - \tan \phi_i \cos \phi_i) \). \( d_i (\sin \phi_i \cos (\theta_i - \theta_i)) \) is the projection of the line segment connecting the reference sensor and sensor \( i \) onto the \( xy \)-target line. If the altitudes of sensors 1, \ldots, \( n \) are all the same, then \( d_i \tan \phi_i \cos \phi_i \) are the same to each other. In this case, we can see that \( I_{\phi\phi}^3 \) is affected by the spread of \( d_i (\sin \phi_i \cos (\theta_i - \theta_i)) \) values. In other words \( I_{\phi\phi}^3 \) is affected by the spread of the sensors along the direction of the \( xy \)-target line.

5 Conclusion

Under the assumption that the distance to the target is much greater than the between-sensor distances, we derived the approximate Fisher information matrix whose inverse yields the variances of the angle and the range estimators of the target location. Investigation of the relationship between the Fisher information and the sensor formation is summarized below.

(i) When all the sensors and the target are on the same plane, the variances of the range and the angle estimators depend on the sensor formation only through the perpendicular distances from the sensors to line of the target and the reference sensor.

(ii) The variance of the range estimates is proportional to the fourth power of the distance to the target.

(iii) The variances of the angle estimates, whether the sensors and the target are on a plane or in a 3 dimensional space, are not dependent on the distance to the target. In particular, the accuracy increases in estimating the azimuthal angle \( \theta_i \) of the target as the sensors are spread more widely in the \( xy \) plane in the direction perpendicular to the \( xy \)-target line. As for the elevation angle \( \phi_i \) of the target, the estimation accuracy increases as the sensors are spread more widely in the \( xy \)-plane in the direction along the \( xy \)-target line.

We included an experimental result of a sensor-formation problem in a 2-dimensional space. But as far as the sensor formation problem is concerned, we need further exploration in the context of sensor array to achieve a maximum accuracy for the estimation of the target location under a variety of conditions of the sensors. The conditions may concern measurement error and the total number of sensors among others. The problem deserves our attention as a future work for improvement of a statistical method for target tracking.

References


