CABLE-HARNESSSED SPACE STRUCTURES: A BEAM-CABLE APPROACH

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ABSTRACT
A homogenization method for deriving an analytical con- tinuum model is presented for determining the vibration frequencies of cable-harnessed beam structures with periodic wrapping patterns. The cable is assumed to exhibit a beam-like behaviour and the potential and kinetic energies of the structure can be formulated. The partial differential equation for the transverse displacement of the structure is obtained by applying Hamilton’s principle and the method of separation of variables is used to calculate the frequencies of vibration. Numerical results are presented for varying system parameters. The introduction of the cable causes a stiffening effect on the system which in turn causes an upward shift in the natural frequencies when compared to an Euler-Bernoulli beam model that only takes into account the additional mass of the cable.

KEY WORDS
Vibration analysis, natural frequencies, homogenization, cable-harnessed structure

1 Introduction

The dynamics of a simple beam are well studied and are used in many different applications. However, the addition of even a single cable to such systems results in much more complicated behaviour which poses a multitude of challenges with respect to modelling. Obtaining accurate models for cable-harnessed beam structures is of great importance for the aerospace industry where structural elements are becoming more lightweight and increasingly wrapped with power and signal cables for control applications.

Refs. [1], [2] and [3] performed initial top-level research on cable-harnessed space structures that included cables attached along beams. These tests concluded that the cables had a significant impact on the dynamics of the system and caused a shift in the frequencies. This work addressed the need for more sophisticated models to predict the behaviour of such composite systems and revealed a scarcity of research in this area. Additional research on the dynamics and parameter estimation of cables was presented in refs. [4], [5] and [6] and it was concluded that the typical cables used for aerospace applications could be described by beam models.

2 Homogenized Continuum Model

In this paper we consider beams with a rectangular cross section and two periodic wrapping patterns. The first wrapping pattern, referred to as diagonal wrapping, involves the cable running diagonally along the top and then around the other three sides of the beam. A single repeated element with the diagonal wrapping pattern is depicted in figure 1. The second wrapping pattern, referred to as zigzag wrapping, involves the cable running diagonally along the top, down the side, diagonally along the bottom forming the same angle as the initial diagonal section, and up the side of the beam. A single repeated element with the zigzag wrapping pattern is depicted in figure 2.

Figure 1. Diagonal wrapping pattern. A single repeated element.
Suppose the beam has height $h$, width $b$, total length $l$, and that a single repeated element has length $L$. It is assumed that the cable is attached at every point along the beam and that a pre-tension $T$ is applied. Assuming an Euler-Bernoulli beam model, imposing a small transverse displacement $z = w(x, t)$ along the beam’s midline results in the following displacement field:

$$u_x = -zw_x(x, t)$$

$$u_y = 0$$

$$u_z = w(x, t)$$

The comma is used to denote partial derivatives. For the given displacement field, the only non-zero strain element is $\epsilon_{xx} = -zw_{xx}$. Due to the transverse bending, the cable will contribute strain energy and kinetic energy to the cable-harnessed system. To include these effects into a continuum model, a homogenization technique is used. This method requires that the total strain energy and the total kinetic energy of a single repeated element be calculated.

The pre-tension as well as the bending of the diagonal sections of cable will cause strain energy in the system. To make the problem more tractable, we will assume that the cable can be described using a beam model that can only undergo compression and tension. Under this assumption, only strain along the cable will contribute to the strain energy. Assuming the cable forms an angle $\theta$ with respect to the $x$-axis (as shown in figures 1 and 2), the strain along the cable due to bending is $-zw_{xx} \cos^2(\theta)$. Combining this with the strain due to the pre-tension, a final expression for the strain along the diagonal cable is determined.

$$\epsilon_c = \frac{T}{E_cA_c} - zw_{xx} \cos^2(\theta)$$

$E_c$ is the Young’s modulus of the cable and $A_c$ is the cross sectional area of the cable. Let $r_c$ denote the radius of the cable, then $A_c = \pi r_c^2$. For the sections of cable that run either vertically or horizontally across the beam, the strain along the cable is simply $\epsilon_c = T/E_cA_c$. The total strain energy in a single repeated element is computed by:

$$U_e = \frac{E_b}{2} \int_{\text{beam}} \epsilon_{xx}^2 \, dV + \frac{E_cA_c}{2} \int_{\text{cable}} \epsilon_c^2 \, dl \quad (1)$$

$E_b$ is the Young’s modulus of the beam and the appropriate expression for $\epsilon_c$ is selected based on the cable length in consideration.

With respect to the kinetic energy, we assume that the velocity in the $x$ direction is negligible and hence is omitted from the analysis. We assume that only the velocity in the $z$ direction contributes to the kinetic energy. Since the displacement field due to imposing $z = w(x, t)$ is known, the total kinetic energy in a single repeated element is computed by:

$$T_e = \frac{\rho_b}{2} \int_{\text{beam}} u_{z,t}^2 \, dV + \frac{\rho_cA_c}{2} \int_{\text{cable}} u_{z,t}^2 \, dl \quad (2)$$

$\rho_b$ and $\rho_c$ is the density of the beam and the cable, respectively.

To solve the integrals in equations (1) and (2), additional assumptions are placed on the system. Firstly, we assume that the curvature of a single repeated element is constant and is equal to the curvature at the centre of the element. Secondly, we assume that the velocity in the $z$ direction, $w_z(x, t)$, does not vary across the length of a repeated element and is equal to the velocity at the centre of the element.

Using these additional assumptions, the integrals in equations (1) and (2) are found and expressions for $U_e$ and $T_e$ are calculated. The homogenization technique involves taking these quantities and dividing by the length of the repeated element to obtain a strain energy density and a kinetic energy density for the cable-harnessed system. In finding these densities, we go from working with a discrete system, since $U_e$ and $T_e$ were only in terms of quantities evaluated at the centre of the repeated element, to working with a continuous system. Integrating these densities over the entire length of the system, consisting of multiple repeated elements, we obtain the general forms for the total strain energy and the total kinetic energy of the system.

$$U_{\text{tot}} = \int_0^l (C_1w_{xx}^2 + C_2w_{x,x} + C_3) \, dx \quad (3)$$

$$T_{\text{tot}} = \int_0^l K_1w_z^2 \, dx \quad (4)$$

The constants $C_i$ and $K_1$ are determined based on the wrapping pattern that is in consideration.

### 2.1 Diagonal Wrapping Pattern

For the diagonal wrapping pattern, the diagonal cable section always has position $z = h/2$ and the length of the repeated element is $L = b/ \tan(\theta)$. Performing the homogenization technique, the constants obtained for the total strain and the total kinetic energy are:

$$C_1 = \frac{E_bI_b}{2} + \frac{E_cA_c h^2 \cos^3(\theta)}{8}$$

$$C_2 = -Th \cos(\theta)$$

$$C_3 = \frac{T^2 \tan(\theta)}{2bE_c A_c} \left( \sec(\theta) + b + 2h \right)$$

$$K_1 = \frac{\rho_b A_b}{2} + \frac{\rho_c A_c \tan(\theta)}{2b} \left( \sec(\theta) + b + 2h \right)$$
\( I_b = bh^3/12 \) is the area moment of inertia for the beam and \( A_b = bh \) is the cross sectional area of the beam.

### 2.2 Zigzag Wrapping Pattern

For the zigzag wrapping pattern, the diagonal cable section has position \( z = h/2 \) for \( 0 \leq x \leq L/2 \) and \( z = -h/2 \) for \( L/2 \leq x \leq L \). The length of the repeated element is \( L = 2h/\tan(\theta) \). Performing the homogenization technique, the constants obtained for the total strain and the total kinetic energy are:

\[
C_1 = \frac{E_h I_b}{2} + \frac{E_c A_c h^2 \cos^3(\theta)}{8}, \\
C_2 = 0, \\
C_3 = \frac{T^2 \tan(\theta)}{2b E_c A_c} \left( \frac{b}{\sin(\theta)} + h \right), \\
K_1 = \frac{\rho_b A_b}{2} + \frac{\rho_c A_c \tan(\theta)}{2b} \left( \frac{b}{\sin(\theta)} + h \right)
\]

### 3 Analysis

#### 3.1 Equation of Motion

The Lagrangian of the cable-harnessed system is found by subtracting the total strain energy, equation (3), from the total kinetic energy, equation (4). Applying Hamilton’s principle, the partial differential equation (PDE) for the transverse displacement of the system is:

\[
C_1 w_{xxxx} + K_1 w_{tt} = 0 
\]  
(5)

The boundary conditions associated with equation (5) are:

- \( w(0, t) \) prescribed or \( w_{xx}(0, t) = 0 \)
- \( w_x(0, t) \) prescribed or \( w_{xx}(0, t) = -\frac{C_2}{2C_1} \)
- \( w(l, t) \) prescribed or \( w_{xx}(l, t) = 0 \)
- \( w_x(l, t) \) prescribed or \( w_{xx}(l, t) = -\frac{C_2}{2C_1} \)

We remark that there exists a non-zero natural boundary condition when the slope at either end of the system is free and that this corresponds to a non-zero moment in the beam. Since \( C_2 = 0 \) for the zigzag wrapping pattern but is non-zero for the diagonal wrapping pattern, we note that having the diagonal string section on a single side of the beam causes a moment in the beam.

#### 3.2 Vibration Analysis

Prior to determining the natural frequencies of the cable-harnessed system, the PDE is reformulated to have homogeneous boundary conditions. Introduce a time-independent solution \( w_E(x) \) that satisfies equation (5) and the associated boundary conditions. Letting \( w(x, t) = w_E(x) + v(x, t) \), the new PDE that must be solved is

\[
C_1 v_{xxxx} + K_1 v_{tt} = 0 
\]  
(6)

The boundary conditions associated with equation (6) are:

- \( v(0, t) = 0 \) if \( w(0, t) \) prescribed or \( v_{xx}(0, t) = 0 \)
- \( v_x(0, t) = 0 \) if \( w_x(0, t) \) prescribed or \( v_{xx}(0, t) = 0 \)
- \( v(l, t) = 0 \) if \( w(l, t) \) prescribed or \( v_{xx}(l, t) = 0 \)
- \( v_x(l, t) = 0 \) if \( w_x(l, t) \) prescribed or \( v_{xx}(l, t) = 0 \)

Since \( w_E(x) \) does not depend on time, the natural frequencies obtained by solving for \( v(x, t) \) will be the same as the natural frequencies obtained by solving for \( w(x, t) \). Using the method of separation of variables, we assume a solution of the form \( v(x, t) = f(x) \cos(\omega t) \). The natural frequencies for the system are found by applying the appropriate boundary conditions and determining the values of \( \omega \) such that \( f(x) \) will have a non trivial solution.

Using this vibration analysis technique results in the natural frequencies being given by \( \omega_i = \mu_i \sqrt{C_1/K_1} \) where \( \mu_i \) depends on the boundary conditions which are chosen and is typically solved numerically.

### 4 Numerical Simulation

Numerical simulations that determine the percentage increase of the frequencies of the cable-harnessed model derived in this paper when compared to the frequencies of an Euler-Bernoulli beam model with mass per unit length that has been modified to include the mass of the cable are presented. For the Euler-Bernoulli beam, the natural frequencies are of the form \( \mu_i \sqrt{E_b I_b/m} \), where \( m \) is the mass per unit length and is updated to include the mass of the cable. Note that the value for \( m \) is calculated by assuming the mass of the cable is uniformly distributed across the system and this produces a constant value resulting in \( m = 2K_1 \). The coefficients \( \mu_i \) are the same for both the homogenized model as well as the Euler-Bernoulli model with updated mass per unit length due to equation (6) being similar in form to the PDE for transverse bending for a homogeneous Euler-Bernoulli beam. For the \( i \)th mode, we are determining:

\[
\Delta \omega_i = \left( \frac{\mu_i^2 \sqrt{C_1/K_1} - \mu_i^2 \sqrt{E_b I_b/m}}{\mu_i^2 \sqrt{E_b I_b/m}} \right) \times 100 \\
= \left( \frac{\sqrt{C_1/K_1} - \sqrt{E_b I_b/m}}{\sqrt{E_b I_b/m}} \right) \times 100
\]

We observe that the percentage increase in the frequencies does not depend on \( \mu_i \) and so we will now denote the increase in frequency by \( \Delta \omega \). The independence of \( \Delta \omega \) on \( \mu_i \) indicates that the increase does not depend on the mode which is being considered or the boundary conditions.
By varying certain system parameters, we investigate the effect on the increase in the frequency of the cable-harnessed system from the Euler-Bernoulli beam with an updated mass per unit length. From this data, we quantify the effect of the variations and any observable differences between the two wrapping patterns. Unless otherwise specified, we present results for an aluminium beam and a copper cable with the following system parameters:

\[
\begin{align*}
E_b &= 6.9 \times 10^{10} \text{ N/m}^2 \\
E_c &= 1.17 \times 10^{11} \text{ N/m}^2 \\
b &= 0.02 \text{ m} \\
h &= 0.003 \text{ m} \\
l &= 0.6 \text{ m} \\
r_c &= 0.001 \text{ m} \\
\rho_b &= 2700 \text{ kg/m}^3 \\
\rho_c &= 8960 \text{ kg/m}^3 \\
N &= 30 \text{ repeated elements}
\end{align*}
\]

We consider the effect of variable cable density, cable modulus, cable radius, and the number of repeated elements. The angle \( \theta \) for each wrapping pattern is determined based on the number of repeated elements in the system.

For varying cable density, figure 3, we see that \( \Delta \omega \) is constant. This reflects the fact that a change in the density of the cable does not affect the strain energy introduced into the system by the cable and the change in cable density affects \( K1 \) and \( m \) in a similar manner. Additionally, we notice that for a constant number of repeated elements, \( \Delta \omega \) is greater for the diagonal wrapping pattern than for the zigzag wrapping pattern.

For varying cable Young’s modulus, figure 4, we see that \( \Delta \omega \) follows a linear nature such that an increase in the cable Young’s modulus causes a larger \( \Delta \omega \). This behaviour is a result of the cable Young’s modulus only affecting the strain energy of the system and having no effect on the kinetic energy. Furthermore, for a constant number of repeated elements, \( \Delta \omega \) is greater for the diagonal wrapping pattern than for the zigzag wrapping pattern.

For varying cable radius, figure 5, we observe that \( \Delta \omega \) follows a roughly quadratic nature and an increase in the cable radius causes a larger \( \Delta \omega \). This behaviour indicates that an increase in the radius of the cable causes a larger increase in the \( C_1 \) term, corresponding to the strain energy, than the increase in the \( K_1 \) term, corresponding to the kinetic energy. Also, for a constant number of repeated elements, \( \Delta \omega \) is greater for the diagonal wrapping pattern than for the zigzag wrapping pattern.

For varying number of repeated elements, figure 6, we notice that \( \Delta \omega \) exhibits a non-linear pattern and that an increase in the number of repeated elements causes a diminishing \( \Delta \omega \). These results demonstrate that the additional cable mass added from the increase in the number of repeated elements is more dominant than the additional
strain energy introduced. The diagonal wrapping pattern produces a larger $\Delta \omega$ than the zigzag wrapping pattern.

Overall, we see that for the system parameters that were varied the diagonal wrapping pattern always produced a larger $\Delta \omega$ than the zigzag wrapping pattern. For the system parameters considered, $\Delta \omega$ was typically in the 1–8% range, except for the case of varying cable radius where $\Delta \omega$ was very large and sensitive to changes.

5 Conclusion

In this paper, we demonstrated the ineffectiveness of only updating the mass per unit length of an Euler-Bernoulli model to include the mass of the cable as a method for modeling cable-harnessed systems. By including the stiffness effects of the cables, the difference between the frequencies obtained for the cable-harnessed system and a mass per unit length updated Euler-Bernoulli beam model were significant, particularly with respect to the radius of the cable. Furthermore, we showed that the wrapping pattern of the cable will have an effect on the magnitude of this increase, but not necessarily on the manner that the system responds (constant, linear, or non-linear relationship).

This shift in the frequencies would be of great interest for vibration control. In such application, vibration suppression of the system is the goal, particularly around the fundamental frequency, which is most easily excited. Using an inaccurate model to determine the fundamental frequency of the system, or other frequencies for that matter, would cause the control objectives to not be met and harmful vibratory motion may occur.

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References