SPECTRUM RECONSTRUCTION FROM RECURRENT NONUNIFORM SAMPLING WITH KNOWN NONUNIFORM SAMPLING RATIOS

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ABSTRACT
In this paper, reconstruction of spectrum from recurrent nonuniform sampling using FFT is described. Nonuniform sampling occurs in a very high-speed waveform digitizing system with interleaved A/D converters. We developed a reconstruction algorithm by deriving the relationship between the DFT of a uniformly sampled signal and the DFT of a nonuniformly sampled signal. Some spectra were reconstructed using the proposed algorithm and the results are compared to two existing algorithms: one uses an alternative DFT and the other uses a filter bank technique. The proposed method performed well at lower computational complexity.

KEY WORDS
Recurrent nonuniform sampling, spectrum reconstruction.

1. Introduction
In this paper, reconstruction of a spectrum from recurrent nonuniform sampling using FFT is described. Recurrent nonuniform sampling occurs in a very high-speed waveform digitizing system with interleaved A/D converters [1]–[4].

As shown in Fig. 1, to increase the sampling frequency \( N \) A/D converters are used. Sampling frequency of each A/D converter is \( 1/NT \) [Hz] and the resulting sampling frequency of the high-speed waveform digitizer is \( 1/T \) [Hz]. Ideally each delay is exactly \( T \) seconds. However, due to the imperfection of the delay, the actual cumulative delay is given by \((n+r_n)T\) where \( r_n \) are termed the nonuniform sampling ratios and should be zero for uniform sampling. In general \( r_0 \) can be assumed to be zero. This system results in recurrent nonuniform sampling. Recurrent nonuniform sampling means that a continuous-time signal is sampled nonuniformly with a periodic pattern as shown in Fig. 2 when 3 A/D converters are used.

A continuous-time signal \( x(t) \) is nonuniformly sampled at

\[
(kN + n)T + r_nT = (kN + n + r_n)T
\]  

where \( k \) in general goes from \(-\infty\) to \( \infty \), \( n \) ranges from 0 to \( N-1 \), \( T \) is the average sampling interval, and \( r_n \) are the nonuniform sampling ratios. For example, \( N = 3 \) in Fig. 2.

Recurrent nonuniform sampling is described in the literature [1]–[8]. In particular, an algorithm to reconstruct a digital spectrum from a nonuniformly sampled signal is proposed in [4] where alternative transform which requires high computational complexity is used. In [6] DFT modulated filter bank is used to reconstruct uniformly sampled signals and the reconstruction of digital spectra are obtained from the signal. 74 coefficients are used for each FIR LPF with cutoff
frequency \( \omega_0 = \pi/N \). This requires very high computational complexity. In this paper, we derived a relationship between the DFT of a uniformly sampled signal and the DFT of a nonuniformly sampled signal. Using the relationship reconstruction of spectrum is estimated from the nonuniformly sampled signal. The experiment verified that the proposed method showed similar performance at much lower computational complexity compared to two aforementioned existing methods.

The paper is organized as follows. In section 2, a relationship between the DFT of a uniformly sampled signal and the DFT of a nonuniformly sampled signal is derived and the new algorithm that can take an advantage of FFT is described. In section 3, experimental comparisons between the proposed and two existing methods are made in terms of performance and computational complexity. Finally, a conclusion is made in section 4.

2. Proposed Reconstruction Method Using FFT

The discrete-time Fourier transform (DTFT) of a uniformly sampled sequence, \( x[kN+n] = x(kNT+nT) \), is

\[
X(\theta) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{N-1} x[kN+n] e^{-j\theta(kN+n)}
\]

where \( \theta \) is termed the digital frequency (or normalized frequency) in radians. The DTFT of the nonuniformly sampled sequence is

\[
\tilde{X}(\theta) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{N-1} \tilde{x}[kN+n] e^{-j\theta(kN+n)}
\]

where the nonuniformly sampled sequence is expressed as

\[
\tilde{x}[kN+n] = x(kNT+nT + r_nT).
\]

Using the inverse DTFT formula the DTFT of the nonuniformly sampled sequence becomes

\[
\tilde{X}(\theta) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{N-1} \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda) e^{j(kN+n)\lambda} d\lambda \right] e^{-j\theta(kN+n)}
\]

where \( X(\lambda) \) is the DTFT of the uniformly sampled sequence as in (2) except that \( \theta \) is replaced by \( \lambda \). By changing the order of the summations and manipulating exponents,

\[
\tilde{X}(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda) \left[ \sum_{k=-\infty}^{\infty} e^{j(k\theta-\lambda)N} \right] e^{j(k\theta-\lambda)N} d\lambda = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} e^{j(k\lambda)N} \tilde{X}(\theta) \]

An impulse train function of \( q \) with the period \( Q \) can be expressed as it own Fourier series so that

\[
\sum_{k=-\infty}^{\infty} \delta(q - kQ) = \frac{1}{Q} \sum_{k=-\infty}^{\infty} e^{j2\pi kQ/\pi}
\]

where \( \delta(q) \) is the unit impulse function.

By substituting \( Q = 2\pi/N \), one obtains

\[
\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(q - k2\pi/N) = \sum_{k=-\infty}^{\infty} e^{j\lambda q}
\]

By replacing \( q \) in equation (8) with \((\lambda-\theta)\) and plugging it into equation (6) one obtains

\[
\tilde{X}(\theta) = \sum_{\lambda=0}^{N-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda) \left\{ \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\lambda - \theta - k2\pi/N) \right\} e^{j\lambda q} e^{j\lambda \theta} d\lambda
\]

Let us assume that a particular frequency \( \theta_0 \) is inside the interval \((0, 2\pi/N)\). Using the sifting property equation (9) becomes

\[
\tilde{X}(\theta_0) = \frac{1}{N} \sum_{k=-N}^{N-1} \sum_{n=0}^{N-1} X(\theta_0 + k2\pi/N) e^{j\lambda q} e^{j\lambda \theta_0} = \sum_{n=0}^{N-1} \sum_{k=-N}^{N-1} \left[ \frac{1}{N} \sum_{\lambda=0}^{N-1} e^{j\lambda q} e^{j\lambda \theta_0} \right] X(\theta_0 + k2\pi/N)
\]

Let us consider the frequency \( \theta_0 + 2\pi/N \). With this frequency equation (10) becomes

\[
\tilde{X}(\theta_0 + 2\pi/N) = \sum_{k=-N}^{N-1} \left[ \frac{1}{N} \sum_{n=0}^{N-1} e^{j(\theta_0+k2\pi/N)q} e^{j(\theta_0+k2\pi/N)\theta_0} \right] X(\theta_0 + k2\pi/N)
\]

In general, for \( m = 0 \) to \( N-1 \),

\[
\tilde{X}(\theta_0 + m2\pi/N) = \sum_{k=-N}^{N-1} \left[ \frac{1}{N} \sum_{n=0}^{N-1} e^{j(\theta_0+k2\pi/N)q} e^{j(\theta_0+k2\pi/N)\theta_0} \right] X(\theta_0 + k2\pi/N)
\]
Equation (12) can be rewritten as
\[ \tilde{X}(\theta_0 + m \frac{2\pi}{N}) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \sum_{n=0}^{N-1} \frac{1}{N} e^{j\left(\frac{nk+2\pi}{N}\right)n} e^{j\frac{2\pi}{N}(m-k)N} X\left(\theta_0 + k \frac{2\pi}{N}\right) \] (13)

Let us define the following.
\[ B(k, m) = \sum_{n=0}^{N-1} \frac{1}{N} e^{j\left(\frac{nk+2\pi}{N}\right)n} e^{j\frac{2\pi}{N}(m-k)N} \] (14)

In other words, \( B(k, m) \) for \( m = 0, 1, \ldots, N-1 \) is the DFT of the sequence given by
\[ \left\{\frac{1}{N} e^{j\left(\frac{nk+2\pi}{N}\right)n}, \frac{1}{N} e^{j\left(\frac{nk+2\pi}{N}\right)n}, \ldots, \frac{1}{N} e^{j\left(\frac{nk+2\pi}{N}\right)n}\right\} \] \( (15) \)

where \( k = -N/2, -(N/2-1), \ldots, 0, 1, \ldots, N/2-2, N/2-1 \). Now equation (13) becomes
\[ \tilde{X}(\theta_0 + m \frac{2\pi}{N}) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} B(k, (m-k) \text{mod} N) X\left(\theta_0 + k \frac{2\pi}{N}\right) \] (16)

Equation (16) can be rewritten as
\[ \tilde{X}(\theta_0 + m \frac{2\pi}{N}) = \sum_{l=0}^{N-1} A(l, (m-l) \text{mod} N) X\left(\theta_0 + l \frac{2\pi}{N}\right) \] (17)

where
\[ A(l, m) = \begin{cases} B(l, m) & \text{for } 0 \leq l \leq \frac{N}{2}-1, \\ B(l-N, m) & \text{for } \frac{N}{2} \leq l \leq N-1 \end{cases} \] (18)

In matrix form, equation (17) becomes
\[ \tilde{X} = AX \] (19)

where
\[ X = \begin{bmatrix} X(\theta_0) \\ X(\theta_0 + \frac{2\pi}{N}) \\ \vdots \\ X(\theta_0 + (N-2) \frac{2\pi}{N}) \\ X(\theta_0 + (N-1) \frac{2\pi}{N}) \end{bmatrix}, \quad \tilde{X} = \begin{bmatrix} \tilde{X}(\theta_0) \\ \tilde{X}(\theta_0 + \frac{2\pi}{N}) \\ \vdots \\ \tilde{X}(\theta_0 + (N-2) \frac{2\pi}{N}) \\ \tilde{X}(\theta_0 + (N-1) \frac{2\pi}{N}) \end{bmatrix}, \text{ and} \]
\[ A = \begin{bmatrix} A(0, 0) & A(1, N-1) & \cdots & A(N-2, 2) & A(N-1, 1) \\ A(0, 1) & A(1, 0) & \cdots & A(N-2, 3) & A(N-1, 2) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A(0, N-2) & A(1, N-3) & \cdots & A(N-2, 0) & A(N-1, N-1) \\ A(0, N-1) & A(1, N-2) & \cdots & A(N-2, 1) & A(N-1, 0) \end{bmatrix} \]

The reconstruction of \( X \), \( \tilde{X} \), can be obtained by
\[ \tilde{X} = A^{-1}X. \] (20)

In practice, instead of finding the inverse matrix and multiplying it to the vector, equation (19) is solved using Gaussian elimination. An algorithm to reconstruct the digital spectrum from the nonuniformly sampled signal is as follows.
1) Compute the DFT, \( \tilde{X}(k) \), of the nonuniformly sampled signal using FFT by padding appropriate number of zeros as necessary. Now the number of the DFT coefficients is \( LN \).
2) Do the following with \( q = 0, 1, 2, \ldots, L-1 \).
   i) Let \( \theta_0 = q \frac{2\pi}{LN} \) and compute \( B(k, m) \) using (14).
   ii) Form matrix \( A(q) \) using equations (18) and (19).
   iii) Solve the following for reconstruction using Gaussian elimination.
\[ \begin{bmatrix} \tilde{X}(q) \\ \tilde{X}(L+q) \\ \vdots \\ \tilde{X}((N-1)L+q) \end{bmatrix} = A^{-1}(q) \begin{bmatrix} X(q) \\ \tilde{X}(L+q) \\ \vdots \\ \tilde{X}((N-1)L+q) \end{bmatrix} \] (21)

3. Experimental Results

In [4], an alternative discrete-time Fourier transform, \( \tilde{X}_f(\theta) \), is defined as follows and the procedure similar to one described in the previous section is used for reconstruction of spectrum.
\[ \tilde{X}_f(\theta) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{N-1} x(kNT + nT + r_T)e^{-j(\theta kNT + \theta nT + \theta r_T)} \] (22)

In [4] instead of constructing a new \( N \) by \( N \) matrix equation for each \( q \) as in (21), only one \( N \) by \( N \) inverse matrix is used in all \( q \). However, FFT cannot be used to compute the alternative DFT and that results in heavy computational complexity.

In [6] modulated filter bank is used to reconstruct uniformly sampled signals and the reconstruction of digital spectra are obtained from the signal. 74 coefficients are used for each
FIR LPF. There are $N$ FIR filters. This requires high computational complexity.

### 3.1 Experiment 1

The following continuous-time signal was used for our first experiment [4].

$$x(t) = \sin(0.2\pi t)$$  \hspace{1cm} (23)

The average sampling interval $T = 0.11$ [sec], the number of A/D converters $N = 8$, and the number of samples $LN = 512$ ($L = 64$) in this experiment. The $r_n$'s are randomly generated from the interval $(-0.5, 0.5)$.

The plots of reconstructed, nonuniform, and true spectra are shown in Fig. 3 which is almost identical to one in [4]. TABLE 1 shows the number of complex multiplications required for reconstruction.

<table>
<thead>
<tr>
<th>To compute $\hat{X}(k)$ or $\hat{X}_j(k)$</th>
<th>Proposed method</th>
<th>Method in [4]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{LN}{2} \log_2(LN)$</td>
<td>$(LN)^2$</td>
<td></td>
</tr>
</tbody>
</table>

| To build $N\times N$ matrix | $N\left(\frac{N}{2} \log_2 N\right) + L$ | $2N + N^3$ (for inverse matrix) |
| To reconstruct spectra | $\left(\frac{N^3}{3} + N^2\right)L$ | $N^2L$ |

When $L = 64$ and $N = 8$, the proposed method needs 23,466 complex multiplications while the method in [4] needs 266,768 multiplications. The complexity ratio is about 11.4. The ratio will increase as the number of samples increases (or $L$ increases).

### 3.2 Experiment 2

The following continuous-time signal was used for our second experiment [6].

$$x(t) = \sin(0.1\pi t) + 2\sin(0.751\pi t)$$  \hspace{1cm} (24)

The average sampling interval $T = 1$ [sec], the number of A/D converters $N = 4$, and the number of samples $LN = 512$ ($L = 128$) in this experiment. Three different cases of nonuniform sampling ratios were used in this experiment as shown in TABLE 2.

**TABLE 2.** Nonuniform sampling ratios used for experiment

<table>
<thead>
<tr>
<th></th>
<th>$r_0$</th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$r_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Almost uniform</td>
<td>0</td>
<td>-0.01</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>b) Near uniform</td>
<td>0</td>
<td>-0.05</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>c) Strong nonuniform</td>
<td>0</td>
<td>-0.5</td>
<td>-1.55</td>
<td>0.55</td>
</tr>
</tbody>
</table>

When delays are almost uniform, the reconstructed spectrum closely matches the true spectrum. The true spectrum that coincides with the reconstructed spectrum is omitted in Fig. 4.

When delays are almost uniform, the reconstructed spectrum still matched the true spectrum as shown in Fig. 5. Note that there is strong presence of at least one extraneous peak in the spectrum of the nonuniform samples. It is completely eliminated in the reconstructed spectrum.
Fig. 5. Plots of reconstructed and nonuniform spectra when nearly uniformly distributed delays were used (≈ 5%).

Finally, the reconstructed spectrum obtained from the strongly nonuniformly sampled signal is shown in Fig. 6. It does not look like the true spectrum especially at the valleys. It is worse than one reported in [6]. However, all spurious peaks which are the result of nonuniform sampling were eliminated.

Fig. 6. Plots of reconstructed and nonuniform spectra when strongly nonuniformly distributed delays were used.

It should be noted that the reconstruction is obtained at much lower computational complexity with the proposed method. In [6] FIR filter operation alone needs $76 \times 4 \times 512 = 155,648$ complex multiplications. Additional computation for modulation, demodulation and inverse matrix is required in [6]. The proposed method needs only 9,130 complex multiplications based on TABLE 1.

4. Conclusion

In this paper, we derived a relationship between the DFT of a uniformly sampled signal and the DFT of a nonuniformly sampled signal. Using the relationship an algorithm for reconstruction of spectrum from the nonuniformly sampled signal is developed. The experiment verified that the proposed method showed similar performance at much lower computational complexity compared to two existing methods.

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References


