CONTROLLER DESIGN FOR NONLINEAR DESCRIPTOR SYSTEMS USING PARTICLE SWARM OPTIMIZATION

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ABSTRACT
This paper proposes a control strategy for descriptor non-polynomial systems using particle swarm optimization (PSO). Basic idea of our approach is to extend the existing approaches for estimating the domain of attraction (DOA) of the non-polynomial systems to synthesis of stabilizing control. To do this we derive the stability conditions for estimating the DOA with input magnitude constraints. From these conditions stabilizing controller is obtained by PSO. The proposed strategy can be easily exploited to search for both the stability controller and optimal estimates. Usefulness and validity are demonstrated by numerical simulations.

KEY WORDS
nonlinear descriptor systems, non-polynomial systems, particle swarm optimization, linear matrix inequality, sum of squares

1 Introduction
It is well known that many of mechanical systems are written in nonlinear dynamical equations. Since these equations are derived from Lagrangian or Hamiltonian, dynamics of these systems are written in highly coupled nonlinear equations. To synthesize stabilizing controller of these dynamical systems Lyapunov-Based approaches[1], adaptive control methodologies[2] and geometric approaches[3] have been proposed. However, since these are all analytical approaches, to apply to the real systems some restrict conditions are required, such as matching conditions[4] and integrability conditions[5]. On the other hand in this last decade computational approaches have been extensively developed based on linear matrix inequality (LMI) or semi-definite programming (SDP) approaches through sum of squares (SOS) or square matricial representation (SMR). Especially, these computational approaches have been applied to analysis and controller synthesis problems of the polynomial systems[6]. Many of these researches are based on the LMI approaches. For example, input-to-state stability analysis[7] and estimating DOA for uncertain polynomial systems[8] have been developed. Moreover, polynomial state feedback and observer design methods[9] and stabilizing controller with bounded actuators[10] have been proposed. However, most of the mechanical systems are not polynomial systems but non-polynomial systems. From this point of view SOS decomposition has been used to analyze the stability of the non-polynomial systems[11]. Furthermore, a strategy for estimating the DOA for non-polynomial systems has been proposed by solving polynomial optimization problems[12]. This has been achieved by converting the nonlinear terms to uncertain polynomial linearly affected by parameters constrained in a polytope which is taken into account the worst case remainders in truncated Taylor expansions. Ichihara[13] has developed these techniques to estimate the DOA for descriptor non-polynomial systems and derived a scalar type stability condition for estimating the DOA.

In this study we extend the novel approach developed in [13] to synthesize a stabilizing controller for descriptor non-polynomial systems using PSO. To do this we derive the stability conditions for estimating DOA with input magnitude constraints. From these conditions stabilizing controller is obtained by PSO. The proposed strategy can be easily exploited to search for both the stability controller and optimal estimates. Usefulness and validity are demonstrated by numerical simulations.

Notation:
The notation used in this paper is rather standard. The symbols $O$ and $I$ denote the zero matrix and the identity matrix of proper dimensions, respectively. For a real symmetric matrix $A$, the inequality $A \succeq O$ means that $A$ is positive semidefinite. Similarly, $A \succ O$ indicates that $A$ is positive definite. The relations $A \preceq O$ or $A \prec O$ means that $A$ is negative semidefinite or $A$ is negative definite, respectively. Also, vert $R$ means vertices of a polytope $R$. Finally, in symmetric block matrices, the symbol “*” is used for terms induced by symmetry.

2 Nonlinear Descriptor Systems
In this section, we introduce the nonlinear descriptor systems which are represented by autonomous non-
polynomial nonlinear equations[13].

\[
\begin{align*}
E_0(x(t)) + \sum_{i=1}^{r} F_i(x(t)) g_i(x_{\tau_i}(t)) &= \dot{x}(t) \\
= a_0(x(t)) + \sum_{i=1}^{r} b_i(x(t)) g_i(x_{\tau_i}(t)), \quad x_0 = x(0),
\end{align*}
\]

(1)

where \( x = [x_1, ..., x_n]^T \in \mathbb{R}^n \) is the state, \( x_0 \in \mathbb{R}^n \) is the initial state, \( E_0, F_1, ..., F_r \in \mathbb{R}^{n \times n}, a_1, ..., a_r \in \mathbb{R}^n, \tau_1, ..., \tau_r \in \{1, ..., n\} \) are indexes, and the function \( g_1, ..., g_r \in \mathbb{R} \) are non-polynomials. Also the DOA of the origin is defined as

\[
D = \left\{ x_0 \in \mathbb{R}^n : \lim_{t \to \infty} \phi(t; x_0) = 0 \right\}
\]

(2)

Let \( v \in \mathbb{R} \) be a LF for the origin in (1). For simplicity, he sets \( v = x^T P x \). Then the level set

\[
\mathcal{V}(c) = \{ x \in \mathbb{R}^n : v(x) \leq c \} \setminus \{0_n\}
\]

(3)

is an estimate of \( D \) if \( \dot{v}(x) < 0, \forall x \in \mathcal{V}(c) \).

It is assumed that the first \( k \) derivatives of non-polynomial \( g_i \) are continuous on \( \mathcal{V}_{\tau_i}(c) = \{ x_{\tau_i} \in \mathbb{R} : x \in \mathcal{V}(c) \} \). Then, for each \( x_{\tau_i} \), there exists a parameter \( \theta_i \in \mathcal{R} \) such that \( g_i \) is equivalent to the Taylor expansion up to degree \( k-1 \) and the Lagrange form of the remainder.

\[
g_i(x_{\tau_i}) = h_i(x_{\tau_i}) + \theta_i \frac{x_{\tau_i}^k}{k!}
\]

(4)

where

\[
h_i(x_{\tau_i}) = \left. \sum_{j=0}^{k-1} \frac{d^j g_i(x_{\tau_i})}{dx_{\tau_i}^j} \right|_{x_{\tau_j}=0} \frac{x_{\tau_i}^j}{j!}.
\]

In the form (4), \( \theta_i \) exists in a following interval [12]

\[
\theta_i \in \left[ \underline{\theta}_i(c), \bar{\theta}_i(c) \right],
\]

(5)

where

\[
\underline{\theta}_i \leq \frac{d^k g_i(x_{\tau_i})}{dx_{\tau_i}^k} \leq \bar{\theta}_i, \quad \forall x_{\tau_i} \in \mathcal{V}_{\tau_i}(c).
\]

By using the interval defined above, it is possible to show that \( \underline{\theta}_i \) and \( \bar{\theta}_i \) construct polynomial functions which provide upper and lower bounds on \( g_i \). By using (4), the system (1) is rewritten to the following implicit form.

\[
E \zeta = f(\zeta, \theta), \quad \theta \in \mathcal{R}
\]

(6)

where

\[
E = \begin{bmatrix} I_n & 0_{n \times n} \\ 0_{n \times n} & n \times n \end{bmatrix}, \quad \zeta = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}
\]

\[
f(\zeta, \theta) = \begin{bmatrix} \dot{x} \\ \sum_{i=1}^{r} f_1(\zeta) \end{bmatrix}, \quad \sum_{i=1}^{r} f_1(\zeta) = a_1(x) - E_1(x) \dot{x}, \quad f_2, f_3 = a_2, x - E_2, x \dot{x}
\]

\[
E_1(x) = E_0(x) + \sum_{i=1}^{r} F_i(x) h_i(x_{\tau_i}), \quad E_2, x = F_i(x) \frac{x^k}{k!}.
\]

\[
a_1(x) = a_0(x) + B(x) h(x), \quad a_2, x = b_i(x) \frac{x^k}{k!}.
\]

\[
B = [b_1, ..., b_r], \quad h = [h_1, ..., h_r]^T.
\]

For the system (6), he defines a LF candidate

\[
v^+(x) = \zeta^T (P^+) E \zeta = \zeta^T E^T P^+ \zeta,
\]

(7)

where

\[
P^+ = \begin{bmatrix} P & 0_{n \times n} \\ S & R \end{bmatrix}
\]

Furthermore, he defines

\[
p(\zeta) = 2 \zeta^T (P^+)^T \begin{bmatrix} \dot{x} \\ f_1(\zeta) \end{bmatrix}, \quad q_i(\zeta) = 2 \zeta^T (P^+)^T \begin{bmatrix} 0_n \\ f_{2, i}(\zeta) \end{bmatrix},
\]

\[
q = [q_1, ..., q_r]^T,
\]

Then the following theorem gives conditions on estimating the DOA of (1)

**Theorem 2.1.** [13] \( \mathcal{V}(c) \) is an estimate DOA of the system(I) if there exists a scalar \( c > 0 \), matrices \( P(> O), S \) and \( R \) such that

\[
p(\zeta) + q(\zeta) \theta < 0
\]

\[
\forall (x, \dot{x}, \theta) \in \mathcal{V}(c) \times \mathbb{R}^n \setminus \{0_n\} \times \text{vert } \mathcal{R}
\]

(8)

### 3 Problem Formulation

We consider control problems of nonlinear descriptor systems, and the system is shown in (9)

\[
\begin{align*}
\left( E_0(x(t)) + \sum_{i=1}^{r} F_i(x(t)) g_i(x_{\tau_i}(t)) \right) \dot{x}(t) &= a_0(x(t)) + \sum_{i=1}^{r} b_i(x(t)) g_i(x_{\tau_i}(t)) \\
+ \left( J_0(x(t)) + \sum_{i=1}^{r} J_i(x(t)) g_i(x_{\tau_i}(t)) \right) u(t)
\end{align*}
\]

(9)

where \( x \in \mathbb{R}^n \) is the state, \( x_0 \in \mathbb{R}^n \) is the initial state, \( E_0 \in \mathbb{R}^{n \times n} \), \( F_1, ..., F_r \in \mathbb{R}^{n \times n}, J_0, ..., J_r \in \mathbb{R}^{n \times n} \) are
polynomial matrices, \(a_0, b_1, \ldots, b_r \in \mathbb{R}^n\) are polynomial vectors, \(\tau_1, \ldots, \tau_r \in \{1, \ldots, n\}\) are indexes, \(g_1, \ldots, g_r \in \mathbb{R}\) are non-polynomials, and \(u \in \mathbb{R}^n\) is an input polynomial vector. For the nonlinear descriptor system (9) we consider state feedback control \(u(t) = u(x(t))\). Then we have
\[
\left( E_0(x(t)) + \sum_{i=1}^{r} F_i(x(t)) g_i(x_{\tau_i}(t)) \right) \dot{x}(t) = \hat{a}_0(x(t)) + \sum_{i=1}^{r} \hat{b}_i(x(t)) g_i(x_{\tau_i}(t)) , x_0 = x(0).
\]  
(10)

Where
\[
\hat{a}_0(x(t)) = a_0(x(t)) + J_0(x(t)) u(x(t)) \quad \hat{b}_i(x(t)) = b_i(x(t)) + J_i(x(t)) u(x(t)), i = 1, \ldots, r
\]

Since the system (10) is an autonomous descriptor system similar to the system (1), the DOA can be estimated by the Theorem 2.1. In order to solve the stabilizing problem of the nonlinear descriptor systems we consider the following problem.

**[Problem]** For the descriptor system (10), find the DOA \(\{x|v(x) \leq 1\}\) and a stabilizing state feedback controller
\[
u(x(t)) = K z(x(t)),
\]  
(11)
satisfying the input magnitude constraints \(|u_j| \leq \mu_j, j = 1, \ldots, m, x \in \{x|v(x) \leq 1\}\). Where \(z \in \mathbb{R}^w\) is monomial vector of \(x\), and \(K \in \mathbb{R}^{n \times w}\) is controller gain.

This problem can be solved by applying the Theorem 2.1 and is shown in following corollary.

**Corollary 3.1.** \(\mathcal{V}(c)\) is an estimate DOA of the system (1) and \(u = Kz(x)\) is a stabilizing controller satisfying the input magnitude constraints if there exists a scalar \(c(> 0)\), matrices \(P (> O)\), \(K, S\) and \(R\) such that
\[
\dot{\mathcal{V}}(c) + \mathcal{V}(c)^T \theta < 0
\]
\[
\forall (x, \dot{x}, \theta) \in \mathcal{V}(c) \times \mathbb{R}^n \setminus \{0_n\} \times \text{vert } \mathcal{R}
\]  
(12)

where
\[
\dot{\mathcal{V}}(c) = 2 \mathcal{V}(c)^T (P^\#)^T \begin{bmatrix} \dot{x} \\ \hat{f}_1(\mathcal{V}) \end{bmatrix} \quad \hat{f}_1(\mathcal{V}) = \hat{a}_1(x) - E_1(\mathcal{V}) \dot{x} \quad \hat{f}_{2,i}(\mathcal{V}) = \hat{a}_{2,i}(x) - E_{2,i}(\mathcal{V}) \dot{x}
\]
\[
\hat{a}_1(x) = \hat{a}_0(x) + \hat{B}(x) h(x), \quad \hat{a}_{2,i}(x) = \hat{b}_i(x) \frac{\mathcal{V}}{h_i}
\]
\[
\hat{a}_0(x) = a_0(x) + J_0(x) K z(x)
\]
\[
\hat{b}_i(x) = b_i(x) + J_i(x) K z(x), i = 1, \ldots, r
\]
\[
\hat{B} = [\hat{b}_1, \ldots, \hat{b}_r], \quad h = [h_1, \ldots, h_r]^T,
\]

and
\[
\begin{bmatrix} P \\ e_j^T K \hat{z}(x) \end{bmatrix} > O, \quad \forall x \in \mathcal{X}, j = 1, 2, \ldots, m
\]  
(13)

where, \(e_j\) is \(j\)th column of identity matrix, \(\hat{z}\) satisfies \(z(x) = \hat{z}(x) x\), and \(\mathcal{X}\) is a compact domain.

**Proof.** Since the system (10) denotes the closed loop system and is an autonomous descriptor system similar to the system (1), the conditions (12) can be derived directly from the Theorem 2.1. Input magnitude constraints also can be rewritten as
\[
x \in \{x|v(x) \leq 1\} \Rightarrow |e_j^T K z(x)| < \mu_j, \quad j = 1, \ldots, m.
\]  
(14)

Then the sufficient condition of (14) is represented as
\[
v(x) \geq \frac{|e_j^T K z(x)|^2}{\mu_j^2}.
\]  
(15)

Substituting \(v(x) = x^T P x\) into (15), we have
\[
P > \frac{\tilde{z}(x)^T K^T e_j^T K \hat{z}(x)}{\mu_j^2}
\]  
(16)

By using Schur complement, it can be shown that (16) is equivalent to (13).

\[
\square
\]

## 4 Controller Design using PSO

### 4.1 Particle Swarm Optimization

In this section we synthesize a stabilizing controller of the system (9) from Corollary 3.1. However, since there exist bilinear terms of design parameters in Corollary 3.1, we cannot apply the LMI approaches to our control problem. Therefore, we split the problem between a parameters search problem and a stability analysis problem, and we alternately solve each problem. To search a controller gain and Lyapunov matrix, we use the PSO algorithm which imitates a behavior of bugs band or fish community. To apply the PSO algorithm, all the elements of the controller gain \(K\) are considered to be a particle. We also prepare \(\hat{P}\) which satisfies \(P = \hat{P}^T \hat{P}\) for the positive definiteness of \(P\) and all the elements of \(\hat{P}\) are considered to be a particle. Thus the particle is constructed of elements of \(K\) and \(\hat{P}\). Proposed controller design method using PSO algorithm is shown in Fig. 4.1. Design procedure is as follows:

1. Elements of \(\hat{P}\) and \(K\) are defined as particles. \(n_p\) particles of \(\hat{P}\) and \(K\) are initialized as random numbers or suitable numbers. When the stabilizing controller gains for linearized systems are known, particles of \(K\) should be initialized by these stabilizing gains.

2. Set the maximal generation number \(n_g\).
3. Then the stability of the linearized system is checked by the negativeness of all the eigenvalues \( \lambda(A+BK_i) \), where the linearized system is \( \dot{x} = (A+BK_i)x \), and \( K_i \) is the linear part of the obtained \( K \). If this condition is satisfied, go to the step 4, else go to the step 6.

\[
f_{val} = \lambda_{max} + \pi \tag{17}
\]

where

\[
\lambda_{max} = \max \lambda(A+BK_i).
\]

4. The input constraint is checked by the following inequality.

\[
\phi_{min} : \text{minimize } \phi
\]

5. The stability condition (12) is checked by the sign of \( \psi_{max} \), which is the solution to the following problem.

\[
\psi_{max} = \max_j \psi_{j,min}
\]

\[
\psi_{j,min} : \text{minimize } \psi_j
\]

\[
\text{s.t. } \tilde{p}(\zeta) + \tilde{q}(\zeta)^T \theta < \psi, \forall (x,\theta) \in \mathcal{V}(1) \times \text{vert } \mathcal{R} \tag{20}
\]

If \( \psi_{max} < 0 \) is satisfied, the stability condition (12) is stabilized.

6. Each particle is evaluated by the following cost function.

\[
f_{val} = \begin{cases} 
\lambda_{max} + \pi & (\lambda_{max} \geq 0) \\
\tan^{-1} \phi_{max} + \frac{\pi}{2} & (\lambda_{max} < 0, \phi_{max} > 0) \\
-\frac{1}{\det \tilde{P}^T \tilde{P}} & (\lambda_{max} < 0, \phi_{max} \leq 0, \psi_{min} < 0) \\
\tan^{-1} \psi_{min} & (\text{otherwise})
\end{cases}
\tag{21}
\]

The lower the value of cost function is, the better the performance is. The first equation of (21) gives a lower value when the eigenvalues of the linearized equation are small. The second equation of (21) gives a lower value when the input constraint (13) is likely satisfied. The third equation of (21) gives a lower value when the estimate DOA size is larger. The fourth equation of (21) gives a lower value when the stability condition (12) is likely satisfied.

7. The particle of the next generation is decided by the PSO algorithm with the above evaluation.

8. If the particle is the last one, go to the step 9, else go to the step 3 for the next particle.

9. If the current generation number is greater than \( n_g \), the calculation finishes, else go to the step 2 for the next generation.
4.2 Example

We apply the proposed design method to a nonlinear descriptor system (22).

\[
\begin{cases}
(2 - \cos x_1) \dot{x}_1 = x_1 + \frac{x_1^2}{3} + u \\
\dot{x}_2 = x_1 - 3x_2 + \sin x_2
\end{cases}
\]  

(22)

We set the controller to be

\[u = Kz(x) = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix},\]

(23)

where the controller variables \(k_1, \ldots, k_5\) are defined as the PSO particles. Also, The Lyapunov function is

\[v(x) = x^T \tilde{P} x \]

\[= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} p_1 & p_3 \\ p_2 & p_4 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\]

(24)

where, \(p_1, \ldots, p_4\) are defined as PSO particles. Thus, there are 9 dimensions each particle. Also, we prepare 100 particles initialized with random number. We take place 300 generations calculation with the proposed algorithm. Then, \(K\) and \(\tilde{P}\) are solved as

\[K = \begin{bmatrix} -3.50 & 0.116 & -0.341 & 0.0468 & -0.476 \end{bmatrix},\]

(25)

\[\tilde{P} = \begin{bmatrix} 0.00950 & -0.151 \\ 0.161 & -0.102 \end{bmatrix}.\]

(26)

Thus, the Lyapunov function \(v\) is derived as

\[v(x) = 2.60x_1^2 - 0.332x_1x_2 + 0.0332x_2^2\]

(27)

The closed loop state trajectories using the controller are shown in Fig. 4.2. All the trajectories on the estimate DOA converges to the origin.

5 Conclusion

In this paper we proposed a control strategy for descriptor non-polynomial systems using the PSO. Basic idea of our approach is to extend the existing approaches for estimating the DOA of the non-polynomial systems to synthesis of stabilizing control. To do this we derived the stability conditions for estimating the DOA with input magnitude constraints. Based on these stability conditions, then we synthesized the stabilizing controller by PSO approach. We have been confirmed the usefulness and validity by numerical simulations.

References


