THE FEEDBACK OF A SINGULARLY PERTURBED SYSTEM MODELED BY BOND GRAPHS

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ABSTRACT
A procedure to build a bond graph of a singularly perturbed system with a state feedback is presented. The gains of the fast and slow dynamics of the system in the physical domain by using reduced bond graphs are determined. Also, the junction structures matrices of the slow and fast bond graphs with the corresponding feedback are proposed. Finally, the proposed procedure is applied to a DC motor.

KEY WORDS
Bond graph, Singular perturbations, State feedback, DC motor.

1 Introduction
A fundamental problem in the theory of systems and control is the mathematical modelling of a physical system. The realistic representation of many systems calls for high-order dynamic equations. The presence of some parasitic parameters, such as small time constants, resistances, inductances, capacitances, moments of inertia, and Reynolds number, is often the source for the increased order and stiffness of these systems. The stiffness, attributed to the simulations occurrence of slow and fast phenomena, gives rise to time scales. The systems in which the suppression of a small parameter is responsible for the degeneration (or reduction) of dimension (or order) of the system are labeled as singularly perturbed systems, which a special representation of the general class of time scale systems [1]. There are several references of singular perturbations methods, some of them are [1], [2] and [3].

A model order reduction is effective to render control design problems to a manageable size when using modern control synthesis methods. Also, the controller order reduction is an interesting research area the enables simpler hardware/software controllers to be obtained [2]. Recent progress in the use of singular perturbation and two-time-scale-methods of modeling and design for control system is reviewed in [4].

In [5] singular perturbation methods are used to provide a basis for the analysis of the asymptotic controllability of multivariable linear systems containing small parasitic elements.

Singular perturbation methods used to reduce the design of stabilising feedback controllers with inaccessible states are found in [6]. A two-stage-sliding-mode controller design for a singular perturbation system is investigated in [7]. The problem of output feedback stabilization of linear systems with a singular perturbation model is addressed in [8]. The singular perturbation system with inaccessible states is considered in [9].

We can find some papers applying bond graph to singular perturbations methods. In [10] describes how the bond graph model is a helpful tool for system analysis in the special case of simplifying the modelling of two time scale systems. The fast and slow dynamics of bond graph models can be estimated by determination of causal loop gains. In [11] the notion of a reciprocal system which, with singular perturbations techniques can obtain more accuracy on the fast time scale behavior of the system. In [12] proposes a bond graph methodology to determine the quasi-steady state model for LTI system with singular perturbations by assigning derivative causality to the fast storage elements and the slow storage elements have an integral causality assignment. The analysis of a class of nonlinear systems with singular perturbations in the physical domain is proposed in [13].

In this paper the modelling in bond graph of a singularly perturbed system with the state feedback is proposed. The feedback gains are determined of each reduced bond graph. Hence, a fast reduced bond graph with the feedback is proposed and the quasi steady state model for the slow dynamic with the corresponding feedback is presented. The junction structures of the reduced bond graphs with the feedback are defined.

Thus, the main contribution of this paper is to obtain the feedback gains by using reduced bond graphs for each dynamic.
Section 2 describes the singular perturbation model. The modelling of singularly perturbed systems in the physical domain is described in Section 3. In Section 4, a closed loop system with singular perturbations in a bond graph approach is presented. The proposed methodology is applied to a DC motor in Section 5. Finally, Section 6 gives the conclusions.

2 The Standard Singular Perturbation Model

The singular perturbation model of finite dimensional dynamic systems, extensively studied in the mathematical literature by Tikhonov (1948,1952), Levinson (1950), Vasil’ eva (1963), Wasow (1965), O’ Malley (1971), etc. was also the first model to be used in control and systems.

Linear time invariant models are of interest in local or small signal approximations of more realistic nonlinear models of dynamic systems [1, 2]. Consider a LTI system to study two time scale properties of the following form,

\[
\begin{align*}
\dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1u, \quad x_1 \in \mathbb{R}^n \quad (1) \\
\dot{x}_2 &= A_{21}x_2 + A_{22}x_2 + B_2u, \quad x_2 \in \mathbb{R}^m \quad (2)
\end{align*}
\]

with output

\[y = C_1x_1 + C_2x_2 \quad (3)\]

The slow reduced model is obtained by setting \( \varepsilon = 0 \) in (2) then

\[\dot{x}_2 = -A_{22}^{-1}A_{21}x_1 - A_{22}^{-1}B_2u \quad (4)\]

substituting (4) into (1) we have,

\[\dot{x}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \quad (5)\]

2.1 Composite State-Feedback Control

In the singular perturbation or, more generally, the two-time-scale approach to feedback design, these stiffness properties are taken advantage of by decomposing the original ill-conditioned system into two subsystems in separate time scales. State feedback design may then proceed for each lower-order subsystem, and the results combined to yield a composite state feedback control is required to achieve an asymptotic approximation to the closed loop system performance that would have been obtained had a state feedback controller been designed without the use of singular perturbation methods [2].

It is desired to construct a state-feedback control for the singularly perturbed linear time-invariant system model analyzed in the previous subsection. Preliminary to any separation of slow and fast designs, the system (1), (2) is approximately decomposed into a slow system model with \( n \) small eigenvalues and a fast system model \( m \) large eigenvalues. The \( n\)-th order slow system is [2]

\[
\begin{align*}
\dot{x}_s &= A_0x_s + B_0u_s, \quad x_s(t_0) = x^0 \quad (6) \\
\bar{x}_2 &= -A_{22}^{-1}(A_{21}x_s + B_2u_s) \quad (7)
\end{align*}
\]

where

\[
\begin{align*}
A_0 &= A_{11} - A_{12}A_{22}^{-1}A_{21} \quad (8) \\
B_0 &= B_{1} - A_{12}A_{22}^{-1}B_2 \quad (9)
\end{align*}
\]

and the vectors \( x_s \) and \( u_s \) are the slow parts of the corresponding variables \( x_1 \) and \( u \) in the control system (1), (2). Also, the \( m\)-th order fast system is

\[\dot{x}_f = A_{22}x_f + B_2u_f, \quad x_f(t_0) = x^0 - x_s(t_0) \quad (10)\]

where \( x_f = x - x_s \) and \( u_f = u - u_s \) denote the fast parts of the corresponding variables in (1), (2).

It is appropriate to consider the following decomposition of feedback controls where

\[
\begin{align*}
u_s &= G_0x_s + v_s \quad (11) \\
u_f &= G_2x_f + v_f \quad (12)
\end{align*}
\]

are separately designed for the slow and fast systems (6) and (10). A composite control for the full system (1), (2) might then plausibly be taken as

\[u_s + u_f = G_0x_s + G_2x_f + v_s + v_f \quad (13)\]

However, a realizable composite control requires that the system states \( x_s \) and \( x_f \) be expressed in terms of the actual system states \( x_1 \) and \( x_2 \). This can be achieved by replacing \( x_s \) by \( x_1 \) and \( x_f \) by \( x_2 - x_s \), so that the composite control (13), in view of (7), takes the realizable feedback form [2]

\[
\begin{align*}
u(t) &= G_2\left[x_2 + A_{22}^{-1}(A_{21}x_1 + B_2G_0x_1)\right] + G_0x_1 \\
&= G_1x_1 + G_2x_2 + v_s + v_f \quad (14)
\end{align*}
\]

where

\[G_1 = (I + G_2A_{22}^{-1}B_2)G_0 + G_2A_{22}^{-1}A_{21} \quad (15)\]

In the next section a system with two time scales using a bond graph model is described [12].

3 Modelling in Bond Graph of a Singularly Perturbed System

Consider the following scheme of a Bond Graph model with an Integral causality assignment (BGI) for a system which includes the key vectors of Fig. 1 [12].
In Fig. 1, \((MS_e, MS_f)\), \((C, I)\) and \((R)\) denote the source, the energy storage and the energy dissipation fields, \((D)\) the detector and \((0, 1, TF, GY)\) the junction structure with transformers, \(TF\), and gyrators, \(GY\).

In this junction structure the storage field is divided in two parts: the slow states \(x_1(t) \in \mathbb{R}^n\) and fast states \(x_2(t) \in \mathbb{R}^m\) both state are associated with the energy variables \(p(t)\) for \(I\) elements and \(q(t)\) for \(C\) elements, \(u(t) \in \mathbb{R}^p\) denotes the plant input, \(z_1(t) \in \mathbb{R}^n\) and \(z_2(t) \in \mathbb{R}^m\) co-energy vector for slow and fast variables, respectively, and \(D_{in}(t) \in \mathbb{R}^p\) and \(D_{out}(t) \in \mathbb{R}^f\) are the relationships between the dissipation field and the junction structure. The constitutive relation for the slow dynamics is

\[
z_1 = F_1 x_1 \tag{16}
\]

for the fast dynamics is

\[
z_2 = F_2 x_2 \tag{17}
\]

and for the dissipations elements is

\[
D_{out} = LD_{in} \tag{18}
\]

The junction structure bond graph in an integral causality assignment is [12]:

\[
\begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2 \\
  \dot{D}_{in} \\
  y
\end{bmatrix} =
\begin{bmatrix}
  S_{11}^{11} & S_{12}^{12} & S_{12}^{11} & S_{13}^{13} \\
  S_{21}^{12} & S_{22}^{22} & S_{22}^{21} & S_{23}^{23} \\
  S_{31}^{21} & S_{31}^{22} & S_{32}^{22} & S_{33}^{23} \\
  S_{41}^{31} & S_{41}^{32} & S_{42}^{32} & S_{43}^{33}
\end{bmatrix}
\begin{bmatrix}
  z_1 \\
  z_2 \\
  D_{in} \\
  u
\end{bmatrix}
\]

(19)

The entries of \(S\) take values inside the set \(\{0, \pm 1, \pm k_t, \pm k_y\}\) where \(k_t\) and \(k_y\) are transformer and gyrator modules for the class of nonlinear systems of this paper; these modules are \(p(t)\) and/or \(q(t)\) associated with \(I\) and/or \(C\) elements in integral causality, respectively. Also, the properties \(S_{11}, S_{22}\) and \(S_{22}\) are square skew-symmetric matrices, and \(S_{12}, S_{12}, S_{11}\) and \(S_{12}, S_{12}, S_{11}\) are matrices each other negative transpose, respectively [12].

The matrices for the slow part of the system defined by \(\dot{x}_1 = A_1 x_1 + A_1 x_2 + B_1 u\) are

\[
A_{11} = (S_{11}^{11} + S_{12}^{11} MS_{21}^{11}) F_1 \tag{20}
\]

\[
A_{12} = (S_{12}^{12} + S_{12}^{12} MS_{22}^{12}) F_2 \tag{21}
\]

\[
B_1 = S_{13}^{13} + S_{12}^{13} MS_{23} \tag{22}
\]

and the matrices for the fast part defined by \(\dot{x}_2 = A_{21} x_1 + A_{22} x_2 + B_2 u\) are

\[
A_{21} = F^{-1}_2 (S_{11}^{21} + S_{12}^{21} MS_{21}^{11}) F_1 \tag{23}
\]

\[
A_{22} = F^{-1}_2 (S_{12}^{22} + S_{12}^{22} MS_{22}^{12}) F_2 \tag{24}
\]

\[
B_2 = F^{-1}_2 (S_{13}^{23} + S_{12}^{23} MS_{23}) \tag{25}
\]

where

\[
M = L (I - S_{22}L)^{-1} \tag{26}
\]

Finally, the matrices for the output \(y = C_1 x_1 + C_2 x_2\) are,

\[
C_1 = (S_{31}^{13} + S_{32} MS_{12}^{13}) F_1 \tag{27}
\]

\[
C_2 = (S_{31}^{23} + S_{32} MS_{22}^{12}) F_2 \tag{28}
\]

\[
D = S_{33} + S_{32} MS_{23} \tag{29}
\]

### 3.1 The Quasi-Steady State Model

A scheme to obtain the quasi-steady state model of a system with singular perturbations in the physical domain is shown in Fig. 2. This junction structure proposes to assign derivative causality for the fast storage elements and the slow storage elements have to maintain the integral causality assignment [12]. This new bond graph is called Singularly Perturbed Bond Graph (SPBG) [12].

![Fig. 2. A junction structure to get the quasi-steady state model.](image)

Finally, the new constitutive relations for the dissipation field is

\[
D_{out}^h = L h D_{in}^h \tag{30}
\]
The junction structure of SPBG of Fig. 2 is given by [12]

\[
\begin{bmatrix}
  \dot{z}_1 \\
  \dot{z}_2 \\
  \frac{D_{in}^f}{z_2}
\end{bmatrix}
= 
\begin{bmatrix}
  H_{11}^{11} & H_{12}^{11} & H_{13}^{11} & H_{14}^{11} \\
  H_{21}^{11} & H_{22}^{11} & H_{23}^{11} & H_{24}^{11} \\
  H_{31}^{11} & H_{32}^{11} & H_{33}^{11} & H_{34}^{11} \\
  H_{41}^{11} & H_{42}^{11} & H_{43}^{11} & H_{44}^{11}
\end{bmatrix}
\begin{bmatrix}
  z_1 \\
  x_2 \\
  \frac{D_{out}^f}{u_s}
\end{bmatrix}
\] (31)

The real roots of the form (2) with (31) and (30) are defined by [12]

\[
x_2 = \tilde{A}_{21} x_1 + \tilde{B}_2 u_s
\] (32)

where

\[
\tilde{A}_{21} = F_2^{-1} [H_{21}^{11} + H_{21}^{12} Q H_{21}^{11}] F_1
\] (33)

\[
\tilde{B}_2 = F_2^{-1} [H_{21}^{13} + H_{21}^{12} Q H_{23}]
\] (34)

being

\[
Q = [I - L_h H_{22}(x)]^{-1} L_h
\] (35)

The quasi-steady state model is given by [12]

\[
\dot{\pi}_s = A_0 \pi_s + B_0 u_s
\] (36)

where

\[
A_0 = [H_{11}^{11} + H_{12}^{11} Q H_{11}^{11}] F_1
\] (37)

\[
B_0 = H_{11}^{13} + H_{12}^{11} Q H_{23}
\] (38)

### 3.2 The Fast Reduced Model

For decoupling the fast behaviour, we can apply the following procedure [10]

**Procedure 1.**

The fast reduced bond graph is deduced from the global one by suppressing:

1. All the C or I elements with large modulus.
2. All the R elements causality connected with these C or I elements directly or indirectly through other R elements.
3. All the input sources having no causal connection with the remaining C, I and R elements.

In the next section a closed loop singularly perturbed system in the physical domain is presented.

### 4 The Feedback of a Bond Graph with Singular Perturbations

This section is focused on linear feedback design for deterministic linear time invariant systems containing both slow and fast dynamic phenomena in the physical domain. Normally, any feedback design, like the system it seeks to control, will suffer from the higher dimensionality and ill-conditioning resulting from the interaction of slow and fast dynamic modes.

In order to get the feedback of a singularly perturbed system of the form (14), the following procedure is proposed.

**Procedure 2.**

1. A bond graph of the system of the fast dynamic is obtained.
2. A fast reduced bond graph with the feedback is determined. Let \( G_2 \) be designed such that \( \text{Re} \lambda (A_{22} + B_2 G_2) < 0 \).
3. The SPBG applying the corresponding feedback is obtained. In addition \( G_0 \) is designed such that \( \text{Re} \lambda (A_0 + B_0 G_0) < 0 \).
4. From (14) with (15) the gains of the original bond graph is obtained.

The feedback of the fast and slow dynamics are drawn in the corresponding bond graphs according to Fig. 3.

![Fig. 3 State feedback in bond graph.](image)

**4.1 Junction Structure of the Fast Reduced Bond Graph with the Feedback**

Considering the fast reduced system modelled by a bond graph and applying the state feedback, the junction structure is given by

\[
\begin{bmatrix}
  \dot{z}_f^C \\
  \frac{D_{in}^f}{z_f^C}
\end{bmatrix}
= 
\begin{bmatrix}
  S_{13}^C & S_{12} & S_{13} \\
  S_{21}^C & S_{22} & S_{23}
\end{bmatrix}
\begin{bmatrix}
  z_f^C \\
  D_{out}^f
\end{bmatrix}
\] (39)

with the constitutive relations

\[
z_f^C = F_f x_f^C
\] (40)

\[
D_{out}^f = L_f D_{in}^f
\] (41)
and the state equation with the feedback is
\[
\dot{x}_f^C = A_{22}^C x_f^C + B_2 v_f
\]
where
\[
\begin{align*}
A_{22}^C &= (S_{11}^C + S_{12} M_f S_{21}^C) F_f \\
B_2 &= S_{13} + S_{12} M_f S_{23}
\end{align*}
\]
being
\[
M_f = L_f (I - S_{22} L_f)^{-1}
\]
Note the \(A_{22}^C = A_{22} + B_2 G_2\) where \(G_2\) is the feedback gain of the fast reduced system.

### 4.2 Junction Structure of the Quasi-Steady State Model in Bond Graph with the Feedback

The quasi-steady state model of a system modelled by bond graphs can be obtained in a step by assigning the derivative causality to the fast storage elements, then the state feedback of the slow storage elements by using Fig. 3 can be applied. Hence, the junction structure of this new bond graph of the slow dynamic with the feedback is defined by
\[
\begin{bmatrix}
\dot{x}_s^C \\
\dot{z}_2 \\
\dot{D}_s^h
\end{bmatrix}
= \begin{bmatrix}
(H_{11}^{11})^C & H_{12}^{11} & H_{13}^{11} \\
H_{12}^{21} & H_{12}^{12} & H_{12}^{13} \\
H_{13}^{21} & H_{13}^{12} & H_{13}^{13}
\end{bmatrix}
\begin{bmatrix}
\dot{x}_s^C \\
\dot{z}_2 \\
\dot{D}_s^h
\end{bmatrix}
\]
and the state equation for the slow dynamic with the feedback is
\[
\dot{x}_s^C = A_0^C x_s^C + B_0 v_s
\]
where
\[
A_0^C = \left( (H_{11}^{11})^C + H_{11}^{12} Q (H_{21}^{11})^C \right) F_1
\]
Note that \(A_0^C = A_0 + B_0 G_0\) where \(G_0\) is the feedback gain of the slow reduced system.

Finally, the feedback gain of the original singularly perturbed system defined by (14) is given by
\[
G_1 = \left[ I - G_{2g} F_{2}^{-1} B_2 \right] G_{og} F_{1}^{-1} - G_{2g} F_{2}^{-1} \tilde{A}_{21}
\]
where \(G_{2g}\) and \(G_{og}\) are the gains into the corresponding bond graphs.

The proposed methodology is applied to DC motor in the next section.

### 5 Example

A DC motor scheme is shown in Fig. 4. This example is well known that it is an electromechanical system. The electrical part is formed by the resistance \(R_a\), and inductance \(L_a\); the inertia \(J\), and friction coefficient \(b\), are the elements of the mechanical part of the system.

A bond graph model in an integral causality assignment of the DC motor is shown in Fig. 5.

The key vectors of the bond graph are
\[
\begin{align*}
D_{in} &= \begin{bmatrix} f_2 \\ f_6 \end{bmatrix}; D_{out} = \begin{bmatrix} e_2 \\ e_6 \end{bmatrix}; u = e_1 \\
L &= diag \{ R_a, b \}
\end{align*}
\]
for slow state
\[
x_1 = p_7; x_1 = e_7; z_1 = f_7; F_1 = 1/J
\]
and for fast state
\[
x_2 = p_3; x_2 = e_3; z_2 = f_3; F_1 = 1/L_a
\]

By applying the Procedure 2, the fast reduced bond graph is shown in Fig. 6.

Now, the fast reduced bond graph with the state feedback is shown in Fig. 7.
The junction structure is

\[
\begin{bmatrix}
e_3 \\
f_2 \\
e_1
\end{bmatrix} =
\begin{bmatrix}
G_{2g} & -1 & 1 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
f_3 \\
e_2 \\
e_1
\end{bmatrix}
\]  \hspace{1cm} (53)

and from (43), (44), (45), (52) and (53) the state equation is

\[
L_a e_3^C = (G_{2g} - R_a) p_3^C + L_a e_1
\]  \hspace{1cm} (54)

Then \( G_{2g} \) be designed such that \( \text{Re} \lambda \left( \frac{-b}{J} - \frac{n^2}{JR_a} + \frac{nG_{0g}}{JR_a} \right) < 0 \).

If we assign derivative causality to the fast storage element \( I : L_a \) and the bond graph (SPBG) has correct causality on each elements then \( A_{22} \) is nonsingular, which is shown in Fig. 8.

![Fig. 8. Quasi-steady state model.](image)

The key vectors and the constitutive relation of the dissipation field for the SPBG are

\[
D_{in}^h = \begin{bmatrix} e_2 & f_6 \end{bmatrix}^T; \quad D_{out}^h = \begin{bmatrix} f_2 & e_6 \end{bmatrix}^T
\]

\[L_h = \text{diag} \left\{ \frac{1}{R_a}, b \right\}\]  \hspace{1cm} (55)

Fig. 9 shows the quasi-steady state model with the feedback state.

![Fig. 9. Quasi-steady state model with the feedback.](image)

The junction structure is

\[
\begin{bmatrix}
e_3 \\
f_3 \\
e_2 \\
f_6
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & n & -1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
G_{0g} - n & -1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
f_7 \\
e_3 \\
G_{0g} - n \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]  \hspace{1cm} (56)

From (35), (38), (48), (51), (55) and (56), the state equation in a closed loop system is

\[
e_7^C = \left( -\frac{b}{J} - \frac{n^2}{JR_a} + \frac{nG_{0g}}{JR_a} \right) p_7^C + \frac{n}{R_a} e_1
\]  \hspace{1cm} (57)

Then \( G_{0g} \) be designed such that \( \text{Re} \lambda \left( \frac{-b}{J} - \frac{n^2}{JR_a} + \frac{nG_{0g}}{JR_a} \right) < 0 \).

From (15) the gain \( G_1 \) is determined by

\[G_1 = (1 - L_a G_{2g}) J G_{0g} + \frac{n^2}{J R_a} G_{2g}\]  \hspace{1cm} (58)

Note that the proposed methodology permits to obtain the gain \( G_1 \) based on the gains \( G_0, G_2 \) and the properties of the SPBG without determining the inverse of \( A_{22} \) matrix.

The state feedback of the original bond graph of the system is shown in Fig. 10.

![Fig. 10. State feedback of the DC motor.](image)

The gain \( G_{1g} \) of the bond graph is \( G_{1g} = G_1 \cdot J \).

Thus, the gains of the complete system are obtained by the reduced fast bond graph and the singularly perturbed bond graph.

6 Conclusion

In this paper the feedback gains of a singularly perturbed system represented in a bond graph are determined. In order to get these gains, the reduced bond graphs of fast and slow dynamics with their corresponding feedback are proposed. The junction structures of each bond graph of the closed loop singularly perturbed system are presented.

References


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