SAFETY FACTOR OF WELDED-PLATE BEAMS BASED ON FINITE
ELEMENT LINEAR BUCKLING ANALYSIS

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ABSTRACT

Beams made of welded plates are very common thanks to the optimum combination of weight and strength of structure. One of the most important design criteria of these beams is their safety against lateral buckling and local buckling. Linear buckling analysis by Finite Element Method (FEM) quickly gives the load multiplier factor to produce elastic buckling. This factor could be considered as the safety factor against buckling if membrane compression stresses in the most critical zone remain well below yield stress up to the buckling load. Otherwise, this interpretation could become unsafe because the combined membrane plus bending stresses in critical zones, which are neglected in linear elastic analysis, could exceed the yield stress. A correction procedure must then be used and may be summarized into the following steps: (1) Carry out a linear static FEM analysis followed by a linear buckling analysis for calculating the lowest load multiplier factor « \( f_{EL} \) » to produce elastic buckling of the structure; (2) Identify the most critical zone of the buckling mode 1 and its width “b”; (3) Check the results of static analysis and identify the value « \( \sigma_{meqL} \) » which stands for « membrane equivalent stress linearized over the buckled width »; (4) Calculate the so called « elastic buckling stress » by \( S_{ef} = f_{EL}\sigma_{meqL} \); (5) If \( S_{ef} \) is low enough, accept \( S_{cr} = S_{ef} \) as the critical stress or else correct the critical stress \( S_{cr} \) by a correction procedure to be defined by considering plastic deformation due to bending across the thickness in critical zones prior to buckling; (6) Calculate the safety factor against buckling by standard formula \( f = S_{cr}/\sigma_{meqL} \). In a similar philosophy of Johnson’s empirical formulas for short columns, the summarized procedure is successfully applied in this paper to numerical examples of thin and moderately thick welded-plate beams for correcting the load multiplier factor given by FEM. Numerical example results show that the proposed correction procedure, or a similar one, is a must-do step after obtaining results of linear buckling analysis by FEM, because the multiplier factor given by FEM could be unsafe or even dangerous for some welded-plate beam designs.

KEY WORDS

Welded plate beams, safety factor, linear buckling, load multiplier factor, finite element method, membrane stress, 2nd order stiffness.

1 Introduction

Even if buckling of structures is highly non linear, the linear theory remains widely applied in structure design thanks to available buckling formulas for simple structures and robust numerical methods for solving Eigen values and Eigen vectors in linear buckling of complex structures. Linear buckling analysis by Finite Element Method (FEM) quickly gives the load multiplier factor that would produce elastic buckling. This factor could be considered as the safety factor against buckling if the membrane compressive stresses in the thickness of critical locations still remain sufficiently below yield stress. Otherwise, cares must be taken to correct the load multiplier factor given by elastic buckling analysis.

A complete theory of linear buckling with examples is given in Timoshenko [1]. This theory is widely applied in linear buckling analysis of complex structures using Finite Element Method (FEM), which can be found in many textbooks about FEM such as Cook et al. [2]. Some references can be found about the application of FEM for analyzing particular and advanced buckling cases, such as Papangelis [3], Venkateswara [4] and Miyazaki [5], but previous works about cares to be taken when computing or predicting linear buckling with FEM for structure design have not been found in the literature other than reference [6].

Since stresses given by FEM need interpretation and judgement for avoiding misusing them, the present work proposes a procedure for interpreting stresses in a static analysis followed by a linear elastic buckling analysis by FEM to determine the safety factor of welded-plate beams.

2 Summary of Linear Buckling Analysis by
Finite Element Method

Beams subjected to high loads such as cranes are usually made of welded thin plates; see Fig. 1. Since compressive stresses exist across entire thicknesses over large zones in those structures, buckling must be considered in their design stage. Linear buckling analysis of structures using FEM consists of two steps summarized in sections 2.1 and 2.2 below.
2.1 Linear Static Analysis

The first step consists of building a geometric FE model, specifying boundary conditions, applying loads on the model and solving for all degrees of freedom (DOF) of the model. The computation sequence in static analysis is as follows.

2.1.1 Equilibrium Equations of Elements

This stage calculates the 1st order stiffness matrix \([K_{1e}]\), the body and surface load vectors \([F_{b} + F_{s}]\) according to matrix equation (1), where the subscript “1” designates 1st order (using zero stress state), the subscripts “e, b and s” stand for element, body and surface. The question mark in this equation, “?”, outlines the fact that the DOFs \([D_{e}]\) and the internal forces \([F_{int}\)] are yet unknown. The number of unknowns in \([D_{e}]\) and \([F_{int}\)] is twice the number of equations, so that equations (1) for each element could not be solved alone.

\[
[K_{1e}][D_{e}] = \{ F_{b} + F_{s} \} + \{ F_{int} \} \tag{1}
\]

2.1.2 Equilibrium Equations of the Model

This stage calculates the total 1st order stiffness matrix \([K_{1t}]\), body and surface load vector \([F_{b1} + F_{s1}]\) by assembling the individual element stiffness matrices \([K_{1e}]\) and applied forces \([F_{b} + F_{s}]\) together to form the assembly equations (2), where the subscript “t” designates total (i.e. assembly) and “ext” for external loads. Here, the total number of unknowns in displacement DOF \([D]\) and in external forces \([F_{ext}]\) is exactly equal to the total number of DOF of the model: each unknown displacement DOF in \([D]\) corresponds to a known external force in \([F_{ext}]\) and, vice-versa, each unknown reaction force in \([F_{ext}]\) corresponds to a restrained boundary condition for which an appropriate DOF in \([D]\) is known. It means that the assembly equation (2) can be solved for all remaining DOF \([D]\) and all reaction forces in \([F_{ext}]\).

\[
[K_{1t}][D] = \{ F_{b1} + F_{s1} \} + \{ F_{ext} \} \tag{2}
\]

2.1.3 Post Processing Calculations

At this stage, all DOF \([D]\) are known. If requested, the internal forces \([F_{int}\)] are calculated from equations (1) and some selected or all stresses \([\{\sigma]\) are calculated using appropriate constitutive relationships such as (3), where \([B]\) represents the displacement-strain transformation matrix and \([C]\) represents the stress-strain material matrix, which contains elastic material properties.

\[
\{ \sigma \} = [C][B][D_{e}] \tag{3}
\]

2.2 Linear Buckling Analysis

Since the 2nd order stiffness matrices depend on stresses, they can only be computed after the static analysis.

2.2.1 Equilibrium Equations Taking the 2nd Order Stiffness into Account

Elastic buckling theory stipulates that the 2nd order stiffness is in function of the geometry and the membrane stress \([\{\sigma_{m}\}) but is independent of bending stress across thicknesses such as shown in (4), Timoshenko [1] and Cook [2]. The element 2nd order stiffness matrices \([K_{2e}]\) are computed based on membrane stresses given by static analysis and assembled together to form the total 2nd order stiffness matrix \([K_{2t}]\).

\[
[K_{2t}] = \text{Function of geometry and } \{\sigma_{m}\} \tag{4}
\]

By assuming that the 2nd order stiffness matrix is proportional to stresses, then if the load is multiplied by a factor \(f\), the assembly stiffness equation can be written by (5) in which, the minus sign before \([K_{2t}]\) means “stiffness reduction” instead of “stiffness increase”.

\[
[K_{1t} - f \cdot K_{2t}][D] = \{ F_{b1} + F_{ext} \} \tag{5}
\]

2.2.2 Solving Eigen value and Eigen vector Equations

The purpose of linear buckling analysis is to determine the multiplier factor \(f\) to bring the applied load to the buckling level at which the displacements would indefinitely increase without increasing the load: \(\{\Delta D\) not all zero while \(\{\Delta F\) = \{zeros\}, and the equation (5) applied for incremental DOF \(\{\Delta D\) becomes an Eigen value and Eigenvector equations as follows.

\[
[K_{1t} - f \cdot K_{2t}][\Delta D] = \{\text{zeros}\} \tag{6}
\]
which give either \( \{ \Delta D \} = \{ \text{zeros} \} \) for any arbitrary value of \( f \) or \( \{ \Delta D \} = \{ \text{not zeros} \} \) if the determinant of the matrix \([K_1 - fK_2]\) is null, i.e.

\[
|K_1 - fK_2| = 0
\]  

(7)

Solving equation (7) gives Eigen values \( f_i \), \( i = 1, 2, \ldots, n \), and solving (6) for each Eigen value \( f_i \) gives the buckling mode \( \{ \Delta D_i \} \). The lowest positive value of \( f_i \) is the most important result of a linear buckling analysis and is named as “linear buckling load factor \( f_E \)” where the subscript \( E \) stands for elastic buckling.

3 Numerical Examples of Linear Buckling of Simple Cranes

Standard steel with Young’s modulus \( E = 2E5 \) MPa and yield stress \( S_Y = 250 \) MPa is used for all examples.

Example 1: A short and thin-plate crane with length \( L = 5000 \) mm, height \( h = 600 \) mm, 400 mm wide flanges, 6 mm thick upper flange, 12 mm thick lower flange and 3 mm thick web is subjected to a distributed dead load \( W_1 = 10 \) kN and carries a \( W_2 = 50 \) kN live load applied on lower flange by 4 rollers (see Fig. 1).

Example 2: A two times longer and thicker crane \( (L = 10000 \) mm, 12 mm thick upper flange, 24 mm thick lower flange and 6 mm thick web) with same height and width is subjected to four times heavier loads \( (W_1 = 40 \) kN and \( W_2 = 200 \) kN). The factors 2 and 4 times are discussed in the numerical results, section 3.2.

Two locations for \( W_2 \) are studied: \( W_2 \) at middle of length \( L \) and \( W_2 \) at 600 mm near an end.

3.1 Finite Element Models

Details about material properties, geometry and load of numerical example 1 are shown in Fig. 2. The beam is modeled by surfaces and meshed into about 13 200 nodes and 13 000 shell elements for the short crane and about 24 500 nodes and 24 100 elements for the long crane. The beam is supported at both ends and laterally supported at mid length. Dead load \( W_1 \) is distributed on the top line of web while the live load \( W_2 \) is applied on 4 contact areas between wheels and inferior flanges.

3.2 Numerical Results of Static and Linear Buckling Analyses

The most essential results of static and linear buckling analyses for four cases using ANSYS Workbench 14, [7], are highlighted on lines 1, 2 and 3 of Table 1. Results on lines 7 and 8 are selected just for discussion.

The most essential steps of interpreting the safety factor “\( f_safety \)” of welded-plate beams against buckling, based on FEM results, are highlighted on lines 4, 5 and 6 of Table 1 and explained in the following sections.

Discussion about 2 times longer and thicker beam with 4 times heavier loads: When the load \( W_2 \) is at middle, the bending stress is the most important: the bending moment \( M \) is 8 times greater (proportional to \( L \) and \( W \)); the cross section moment of inertia \( I \) is about doubled (2 times thicker); thus the bending stress \( \sigma = M/c/I \) is about 4 times higher (line 3 of case 3 compared to case 1). But, applying 4 times greater stress on about 4 times greater buckling strength plate (slenderness ratio \( \lambda = b/t \) is doubled) would give about the same safety factor (line 1 of case 3 compared to case 1). When the load \( W_2 \) is near an end, the shear stress in web is the most important: the shear force \( V \) is about 4 times greater (proportional to load) and the web area is doubled; thus the shear stress, approximately given by \( \tau = V/A_{\text{web}} \), is about two times higher. But, applying twice higher shear stress on a four times higher buckling strength of web (slenderness \( \lambda = h/t_w \) is doubled) would be about twice safer (line 1, cases 2 and 4).

3.3 Safety Factor based on FEM Results

The present work proposes a procedure for reasonably interpreting the safety factor of welded-plate beams against buckling based on FEM results. This procedure consists of six following steps.
1. **Linear buckling load factor “f_E”** (line 1 in Table 1): Such as mentioned in sub section 2.2.2, \( f_E \) is the most important result of a linear buckling analysis by FEM. In four studied cases, \( f_E \) values in line 1 mean that the short-and-thin crane would elastically buckle if the loads are multiplied by 2.07 when \( W_2 \) is at middle of length (case 1) and by 2.44 when \( W_2 \) is near an end (case 2), and the long-and-thicker crane would elastically buckle if the loads are multiplied by 2.18 when \( W_2 \) is at mid length (case 3) and by 4.51 when \( W_2 \) is near an end (case 4).

Could these \( f_E \) values (2.07, 2.44, 2.18 and 4.51) be considered as the safety factors against buckling of the cranes? The next steps will answer this question.

2. **Identification of buckling zone and width**: Fig. 3 illustrates the buckling modes of four studied cases.

<table>
<thead>
<tr>
<th>Common data: h = 600 mm, b_f = 400 mm, ( \Delta X_e = 200 \text{ mm}, \Delta Z_e = 120 \text{ mm}, \Delta X_c = 20 \text{ mm}, \Delta Z_c = 50 \text{ mm} )</th>
<th>( L = 5000 \text{ mm}, t_{\text{upper}} = 6 \text{ mm} ) ( t_{\text{lower}} = 12 \text{ mm}, t_w = 3 \text{ mm} )</th>
<th>( L = 10000 \text{ mm}, t_{\text{upper}} = 12 \text{ mm} ) ( t_{\text{lower}} = 24 \text{ mm}, t_w = 6 \text{ mm} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>Case 2</td>
<td>Case 3</td>
</tr>
<tr>
<td>( W_2 ) at middle</td>
<td>( W_2 ) near end</td>
<td>( W_2 ) at middle</td>
</tr>
<tr>
<td>2.06</td>
<td>2.44</td>
<td>2.22</td>
</tr>
<tr>
<td>( \sigma_{\text{eqy}} ) on buckled width (MPA)</td>
<td>40.8</td>
<td>57.7</td>
</tr>
</tbody>
</table>

\[ f_{\text{cr}} = \frac{\sigma_{\text{cr}}}{\sigma_{\text{eqy}}} \]

\[ f_{\text{cr}} \leq \frac{1}{2} S_V \Rightarrow \sigma_{\text{cr}} = f_{\text{cr}} S_V \]

\[ f_{\text{cr}} > \frac{1}{2} S_V \Rightarrow \sigma_{\text{cr}} = \frac{S_V}{1 - 0.25 S_V/S_{\text{cr}}} \]

6. **Safety factor** \( f = \frac{S_{\text{cr}}}{\sigma_{\text{eqy}}} \)

7. \( \sigma_{\text{eqy}} \) maximum on surface (MPA)

8. \( \sigma_{\text{eqy}} \) on buckled width (MPA)

<table>
<thead>
<tr>
<th>( f_{\text{cr}} )</th>
<th>( \sigma_{\text{cr}} )</th>
<th>( \sigma_{\text{eqy}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>84.0</td>
<td>140.8</td>
<td>359.4</td>
</tr>
<tr>
<td>84.0</td>
<td>139.0</td>
<td>206.5</td>
</tr>
<tr>
<td>2.06</td>
<td>2.41</td>
<td>1.28</td>
</tr>
<tr>
<td>-42.2</td>
<td>-46.2</td>
<td>-161.2</td>
</tr>
<tr>
<td>149.2</td>
<td>144.3</td>
<td>199.6</td>
</tr>
</tbody>
</table>
The buckling mode of case 1, Fig. 3(a), shows that the web in the mid length zone is most critical (not the flange) so that the identified buckled width is the height h of the web in mid length zone. The buckling modes of cases 2 and 4 are similar and show that the most critical buckled width is the height h of web in end zone. The buckling mode of case 3 shows that the most critical buckled width is half of the upper flange width (b_f/2) in mid length zone.

The identification of the buckled zone and width is essential for the interpretation of stresses that would have the greatest effect on the buckling of structure, which is presented in the step 3 below.

3. Membrane equivalent stresses linearized on buckled width: Since FEM results are dependent of meshed models and may be singular, i.e. may not converge when refining meshes, they must be linearized across the width in buckled zone to get equivalent membrane plus bending stresses. The linearization of a stress component $\sigma_i$ ($i = x, y, ...$) consists of calculating the resultant force $F_i$ and bending moment $M_i$ across a distance b by two integrals

$$ F_i = \int_{-b/2}^{b/2} \sigma_i \, dy ; \quad M_i = - \int_{-b/2}^{b/2} y \cdot \sigma_i \, dy $$

and

$$ \sigma_{il} = \frac{F_i}{b} - \frac{6 \cdot M_i \cdot y}{b^2} $$

(8)

After linearizing all stress components ($\sigma_x, \sigma_y, ...$), the linearized equivalent stresses are computed by

$$ \sigma_{equiv} = \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x \sigma_y - \sigma_y \sigma_z - \sigma_z \sigma_x + 3(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)} $$

(9)

Furthermore, since the buckling does not depend on bending across plate thicknesses, only membrane stresses across thicknesses, are linearized and interpreted for buckling analysis: see Fig. 4 for $\sigma_{meqL}$, which stands for “membrane equivalent stresses linearized on buckled width”, of four studied cases.

It is noticed that the cases 1, 2 and 4 show singular FEM results in web at junction with upper and lower flanges but the linearized results converge, i.e. practically independent of refining meshes while the case 3 does not show any singularity.

4. Identification of elastic critical stress $S_{crE}$: If the loads are multiplied by the factor $f_E$, all stresses would also be theoretically multiplied by $f_E$ and the structure would elastically and suddenly buckle. In a similar way of single column subjected to axial compression and bending moment for which the elastic critical stress is the membrane stress due to critical axial force, $S_{crE} = F_{cr} / A$, it is proposed for the multi axial stress in welded-plate beams to consider $S_{crE}$ as the maximum membrane equivalent stress on the buckled width, given by (10).

$$ S_{crE} = f_E \cdot \sigma_{meqL} $$

(10)

The numerical results are $S_{crE} = 84$ MPa, $140.8$ MPa, $359.4$ MPa and $529.2$ MPa for cases 1, 2, 3 and 4, respectively (reproduced from line 4 of Table 1).
Some of these results look unreasonable (359.4 and 529.2) and the next step will show how to correct them.

5. Correction of critical stress ($S_{cr}$): In a similar way of buckling of single column, one could introduce a slenderness ratio $\lambda$ of the buckled zone such as (11).

$$\lambda = k(b/t)^n$$

where $b$ and $t$ are the width and the thickness of the buckled zone, and $k$ and $n$ are the constants depending on boundary conditions and stress distribution in the buckled zone, to be defined so as the elastic buckling stress $S_{crE}$ is inversely proportional to $\lambda^2$, such as (12).

$$S_{crE} = \beta \frac{E}{\lambda^2}$$

It is proposed to correct the critical stress in a similar way as Johnson’s parabola shown in Fig. 5 whenever the elastic buckling stress is above half of yield stress for taking into account the plastic deformation due to additional bending stresses prior to buckling, [8].

Fig. 5 - Johnson’s correction for critical stress $S_{cr}$

The linear buckling load factors ($f_2$) of previous four cases and the interpreted safety factors ($f$) are shown below for comparison purposes, reproduced from lines 1 and 6 of Table 1.

<table>
<thead>
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<td>Linear buckling load factor $f_2$</td>
<td>2.06</td>
<td>2.44</td>
<td>2.22</td>
</tr>
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</table>

6. Safety factor against buckling:

$$f_{safety} = S_{cr}/\sigma_{meqvl}.$$  \hspace{1cm} (14)

where again, $\sigma_{meqvl}$ stands for “membrane equivalent stress linearized on width of buckled zone”.

The linear buckling load factors ($f_2$) of previous four cases and the interpreted safety factors ($f$) are shown below for comparison purposes, reproduced from lines 1 and 6 of Table 1.

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</tr>
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</table>

It is noticed that the linear buckling load factor $f_2$ given by FEM could be the safety factor against buckling for thin-plate beams (case 1 and maybe case 2) but carers must be taken for moderately thick plate beams such as cases 3 and 4: the case 3 should be concluded as non satisfactory because $f=1.28$ is not safe enough.

4 Conclusion

Although finite element method is practical nowadays for design of structures, cares must be given to linear buckling analysis when evaluating the safety factor of structure against buckling.

Numerical examples described in section 3 show that a correction procedure is a step an engineer must do after obtaining results of linear buckling analysis by FEM, because the multiplier factor given by FEM could be unsafe or even dangerous for some welded-plate beam designs.

The correction procedure described in section 3.3 is a practical guide for correcting the load multiplier factor given by linear buckling analysis using finite element method with the similar assumption as Johnson’s empirical correction procedure for buckling of single columns.

Since the experimental data about buckling of complex welded-plate structures do not exist due to unrealistic apparatus and costs those experiments would need, there would be difficult to validate buckling FEM results including the correction procedure proposed in this paper.

One of proposed future works on the experimental side is to create data on buckling of reduced size of welded-very-thin-plate structures and even though, the work would be timely and costly due to numerous experiments to be carried out.

One of near future works for validating linear buckling FEM results including the proposed correction procedure, would be to do nonlinear analyses of same structures, independently from linear buckling theory, by creating either eccentric geometry or eccentric loads which are
more complex than but in a similar way of studying single
columns subjected to eccentric loads presented in text
books such as [8].

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