EFFECT OF FEED-THROUGH TERM AND THE SYNTHESIS OF
MULTI-MODE INTEGRAL RESONANT CONTROL

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ABSTRACT
Most damping controller designs are based on, and aimed at, damping the first dominant resonant mode of the targeted system. The higher-order dynamics of these systems are left unmodeled and an adequate feed-through term is generally added to compensate for model truncation effects. The Integral Resonant Control (IRC) exploited this feed-through term to manipulate the dynamics of resonant colocated systems such that a simple integrator delivered adequate damping to the resonant mode. As an added advantage, IRC also imparted damping to other higher-order modes and was guaranteed stable. Nevertheless, this was still a single-mode based design. In this paper, we formalize the effect of the feed-through term on a typical multi-mode system and present the design and analysis of a true multi-mode IRC.

KEY WORDS
Vibration control, Resonance damping, Colocated systems

1 Introduction

Colocated systems are systems where the sensors and actuators are physically place at the same location on the system under test. It has been shown that such systems possess a unique dynamic behaviour where their frequency response exhibits an alternating pattern of complex conjugate poles and zeros. This property was then exploited to formulate the Integral Resonant Control (IRC) technique for effective damping of the lightly damped resonant mode of the system [1]. Over the years, the IRC damping controller strategy has gained substantial popularity due to its simplicity, robustness under parameter uncertainty and guaranteed stability. As such, IRC has been applied to damp the problematic, lightly damped resonant modes of a number of systems such as cantilever beams [1], piezoelectric tube scanners [2], piezo-stack actuated nanopositioners [3], scanning probe microscopes [4], flexible robotic manipulators [5], pedestrian foot bridges [6], large civil structures [7] and many others.

As with other prevalent damping controller designs such as Positive Position Feedback (PPF) [8], Positive Velocity and Position Feedback (PVPF) [9] and Resonant Controller [10], the IRC design is also based on, and designed to target, the first resonant mode of system. Yet, it has been seen during multiple implementations that the IRC is capable of damping multiple resonant modes of any given system. However, the full design and analysis of a truly multi-mode IRC scheme has not been proposed yet. The main hurdle to achieve this goal was the full mathematical understanding of the effect of the feed-through term on the system dynamics. In [3], the full mathematical analysis of the effect of the feed-through term on a second-order colocated transfer function was presented. This paper extends that result further to address the effect of the additive feed-through term on a typical multi-mode system such as a colocated cantilever beam as shown in Figure 1 (a).

The rest of the paper is organized as follows. Section 2 will briefly introduce the dynamic behaviour of typical colocated systems by employing the interconnected mass-spring-damper model and then revisit the conceptual working of the Integral Resonant Control scheme. Section 3 presents the full mathematical analysis of the feed-through term on multi-mode colocated dynamics. A simulation-based design and analysis of multi-mode IRC scheme is presented in section 4 and concluding remarks will be given in section 5.

2 Background

Colocated systems are systems with spatially colocated (at the same position) sensor-actuator pairs. The main advantages of this particular arrangement of sensors and actuators are simplicity in the structural design and greater control authority. In this section, a typical model of colocated systems using interconnected mass-spring-damper model will be presented and key peculiarities in the dynamic behaviour of such systems will be highlighted. The concept of model truncation and feed-through based compensation of the truncation effect will also be revisited. Finally, the Integral Resonant Control scheme will be described briefly.

2.1 Colocated system modeling and their dynamic behaviour

The most generic example of a structure that is popularly employed in the theoretical and experimental analysis
of colocated systems is the cantilever beam incorporating colocated sensor-actuator pairs, as shown in Figure 1. The colocated frequency response measured from output $y$ to input $w$ is shown in Figure 1(b). This can be modeled as an infinite series of mass-spring-damper interconnections as shown in Figure 1(b).

This system can be modeled in the $s$-domain by an infinite sum of second-order transfer functions and can be mathematically written as

$$G(s) = \sum_{n=1}^{\infty} \frac{\sigma_n^2}{s^2 + 2\zeta_n\omega_n s + \omega_n^2},$$

(1)

where $\zeta_n$ is the damping coefficient, $\omega_n$ is the natural (resonant) frequency and $\sigma_n^2$ is the D.C. gain for the $n^{th}$ resonant mode. This transfer function exhibits a pole-zero interlacing throughout its frequency response and this property is of immense interest and utility for designing simple, well-performing damping control techniques.\(^1\) As it is impossible to record the frequency-response of any system for infinite frequencies (and as only a finite bandwidth starting from 0 Hz is generally of interest for any particular application), a truncated model is used, given by Eq. 2.

$$\tilde{G}(s) = \sum_{n=1}^{M} \sum_{\epsilon=1}^{\infty} \frac{\sigma_n^2}{s^2 + 2\zeta_n\omega_n s + \omega_n^2}$$

(2)

Note that in the above model, the number of complex conjugate pole-pairs is $M$ while the number of complex conjugate zero-pairs is $M - 1$. It is well-known that this truncation of the out-of-bandwidth poles has the effect of perturbing the actual position of zeros within the bandwidth of interest\(^{12}\) and to compensate for this, a suitable feed-through term (generally small and real) is added to the truncated $s$-domain model. Thus, the final useful model of a colocated system’s transfer-function takes the form:

$$\tilde{G}(s) = \sum_{n=1}^{\infty} \frac{\sigma_n^2}{s^2 + 2\zeta_n\omega_n s + \omega_n^2} + d,$$

(3)

where $d$ is the feed-through term. In \([1]\), it was shown that this feed-through term could be used to manipulate the position of zeros of the colocated system, the basis for the design of the Integral Resonant Control (IRC) damping technique.

### 2.2 Integral Resonant Control

Integral Resonant Control (IRC) technique is a simple, well-performing closed-loop damping technique capable of delivering substantial damping to multiple resonant modes and behave robustly under changes in system resonances (caused by environmental effects such as ageing, humidity, temperature or due to changes in operating conditions such as loading) \([1]\). The overall concept of the IRC design can be summarized as follows:

*Given that a colocated system with pole-zero interlacing is to be damped, an adequate feed-through term can be added to the system to reverse the interlacing from pole-zero to zero-pole. Furthermore, if a simple integrator with a gain $k$ is implemented in closed-loop with such a modified system, as $k$ increases, the poles of the system (corresponding to the structural resonant modes) traverse a curve where first they move away from the imaginary axis, into the left-half complex plane (thus increasing their damping coefficient) and then back towards the imaginary axis till they reach their correspondingly paired zero (with reduced damping),*
3 Effect of the feed-through term on multi-mode colocated dynamics

A substantial amount of literature exists on how an adequate feed-through term (usually a small real number) is employed to account for model-truncation errors in colocated systems [13, 14]. Yet, the complete characterization of the effect of feed-through term has never been reported. In this section, the overall effect of the feed-through term on the truncated multi-mode system model is expounded upon. These results can easily be applicable to various systems where multiple modes or modes other than the first one need to be damped; for example: overhead cranes, satellite antenna booms, flexible manipulator arms and micro-cantilever sensors. As has been the case historically, all the results in this paper have been derived using systems with no initial damping, to avoid mathematical complexity.

It has already been proved in [15], that the addition of a feed-through term to the truncated model of the colocated system results in an extra pair of zeros, making the number of poles and zeros in the truncated model of the colocated system given by Eq. 3, equal. Mathematically, this can be written as:

\[ G(s) = \sum_{i=1}^{M} \frac{\alpha_i}{s^2 + \omega_i^2} + d, \]  

s.t. \( \omega_i^2 < \omega_{i+1}^2 \) \( \forall i \) and \( d \in \mathbb{R} \) then,

\[ \hat{G}(s) = G(s) + d = (s^2 + \omega_1^2) \sum_{i=1}^{M} \frac{\beta_i}{s^2 + \omega_i^2}. \]

It has been found through experiments as well as simulations that the position of these feed-through-introduced zeros depends on the magnitude and sign of the feed-through term. A mathematical proof of this behaviour is given below.

**Theorem 1** If a feed-through term \( d \) is added to the truncated model of a colocated system given by \( \hat{G}(s) = \sum_{i=1}^{M} \frac{\alpha_i}{s^2 + \omega_i^2} + d \), then the relationship between the feed-through term and the position of the feed-through-introduced zeros \( z_M \) is given by

- \( z_M \in \mathbb{C}, -\omega_{z_M}^2 > -\omega_i^2, \forall i \) if \( d < d_c \), where 
  \[ d_c = -\sum_{i=1}^{M} \frac{\alpha_i}{\omega_i^2} \]
- \( z_M \in \mathbb{C}, -\omega_{z_M}^2 < -\omega_i^2, \forall i \) if \( d > 0 \)
- \( z_M \in \mathbb{R} \) if \( d \in [d_c, 0) \)

**Proof:**

\[ \hat{G}(s) = \sum_{i=1}^{M} \frac{\alpha_i}{s^2 + \omega_i^2} + d = \prod_{i=1}^{M} \frac{(s^2 + \omega_i^2)}{s^2 + \omega_{z_M}^2} \]

which can be rewritten as a ratio of two polynomial:
The roots of the polynomial \( N(s^2) \) denoted by \( (\omega_1^2, \omega_2^2, \ldots, \omega_M^2) \). As \( \omega_i \) are in ascending order, s.t. \( \omega_1^2 < \omega_2^2 < \ldots < \omega_M^2 \), it is clear that

\[
N(-\omega_1^2) = \prod_{j \neq 1} (\omega_j^2 - \omega_1^2) > 0
\]

\[
N(-\omega_2^2) = \prod_{j \neq 2} (\omega_j^2 - \omega_2^2) < 0
\]

This proves that there exists a pair of complex zeros s.t. \( -\omega_2^2 < -\omega_1^2 < -\omega_3^2 \). Continuing this argument, the location of \( M - 1 \) pairs of complex zeros can be written as:

\[-\omega_M^2 < -\omega_{M-1}^2 < \ldots < -\omega_2^2 < -\omega_1^2 < 0.\]

The location of the \( M^{th} \) pair of zeros \( (\omega_M^2) \), can be ascertained by further investigating how the sign of \( N(0) \) and \( D(0) \) changes for various values of \( d \).

\[
N(0) = \sum_{i=1}^{M} (\alpha_i \prod_{j \neq i} \omega_j^2) + d \prod_{i=1}^{M} \omega_i^2
\]

\[
D(0) = \prod_{i=1}^{M} \omega_i^2 > 0
\]

Since \( D(0) > 0 \),

\[
sgn\left(\frac{N(0)}{D(0)}\right) = sgn(N(0))
\]

Therefore,

\[
sgn(N(0)) = sgn\left(\frac{N(0)}{D(0)}\right) = sgn\left(\sum_{i=1}^{M} \frac{\alpha_i}{\omega_i^2} + d\right) \tag{5}
\]

Defining \( d_c = -\sum_{i=1}^{M} \frac{\alpha_i}{\omega_i^2} \), a critical value for the feed-through term \( d \), Eq.5 can be reframed as:

\[
sgn(N(0)) = sgn(d - d_c) \tag{6}
\]

Using Eq.6, three cases based on possible values the feed-through term \( d \) can take, are investigated to find the location of the \( M^{th} \) zero-pair.

- **Case 1:** \( d < d_c < 0 \)

  From Eq.6, \( N(0) < 0 \) and since \( N(-\omega_1^2) > 0 \), it can be concluded that the last root \( (M^{th} \) zero-pair) is in the range of \( (-\omega_1^2, 0) \). Thus,

  \[-\omega_M^2 < -\omega_{M-1}^2 < \ldots < -\omega_2^2 < -\omega_1^2 < \omega_2^2 < 0\quad \text{for } d < d_c.\]

- **Case 2:** \( d > 0 \)

  Calculate \( D(-\infty) \) and find it’s sign

  \[
  D(-\infty) = \prod_{i=1}^{M} (-\infty + \omega_i^2)
  \]

  \[
  D(-\infty) = \begin{cases} 
  > 0 & \text{if } M \text{ is even} \\
  < 0 & \text{if } M \text{ is odd}
  \end{cases} \tag{7}
  \]

  Therefore,
\[ \text{sgn}(N(-\infty)) = (-1)^M \text{sgn}(D(-\infty)) \]
\[ \text{sgn}(N(-\infty)) = (-1)^M \text{sgn}\left(\sum_{i=1}^{M} \frac{\alpha_i}{-\infty + \omega_i^2} + d\right) \]
\[ \text{sgn}(N(-\infty)) = (-1)^M \text{sgn}(d) \]

Since \( d > 0 \), \( \text{sgn}(N(-\infty)) = (-1)^M \); while \( \text{sgn}(N(-\omega_M^2)) = (-1)^{M-1} \). Therefore,

\[ \text{sgn}(N(-\omega_M^2)).\text{sgn}(N(-\infty)) = -1 \quad \text{for} \quad d > 0 \quad (8) \]

Then equation 8 results that:

\( -\infty < -\omega_M^2 < -\omega_1^2 < \ldots < -\omega_2^2 < -\omega_1^2 < 0 \) \quad \text{for} \quad d > 0.

\( \circ \) Case 3: \( d_c \leq d < 0 \)

In this range \( N(0) > 0 \) (see 6). The sign of \( N(s^2) \) when \( s^2 \to \infty \) needs to be found. Following a similar procedure employed to ascertain the sign of \( D(-\infty) \), it can be concluded that:

\[ \text{sgn}(N(\infty)) = \text{sgn}(d) = -1. \]

It is known that the sign of \( D(\infty) = 1 \). Therefore,

\[ \text{sgn}(N(0)).\text{sgn}(N(\infty)) = -1 \quad \text{for} \quad d_c < d < 0 \]

Thus, \( 0 < -\omega_M^2 < \infty \) which means in this range system has two opposite real zeros.

\( \square \)

Note that:

- Adding a feed-through term changes the location of zeros pre-existing in the colocated system transfer function.
- The feed-through term does not move these zeros to a frequency beyond that of their paired pole.
- The additional pair of zeros introduced by the feed-through term does not lie between any two system pole-pairs.

Therefore, there exists only one pair of complex zeros between any two consecutive pole pairs. Figure 4 shows the change in zero location over variations in the feed-through term, for a generic \( M^{th} \)-order colocated system.

As can be seen from related literature cites earlier, colocated positioning of the actuator and the displacement sensor is the most popular arrangement adopted in a number of systems that require damping control. The next section presents simulations for the design and analysis of a multi-mode IRC scheme.

4 Simulation-based design and analysis of multi-mode IRC

The measured colocated magnitude response \( G_{yw}(s) \) for the cantilever beam shown in Figure 1 is given in Figure 2(a). The first three resonant modes as identified in the frequency response can be modeled as a sum of three second-order resonant sections given by:

\[
G_{yw} = \frac{225}{s^2 + 3.854s + 6035} + \frac{8971}{s^2 + 1.49s + 217100} + \frac{90960}{s^2 + 3.573s + 1.697 \times 10^6}.
\]

Using Theorem 1, the critical feed-through term needed to flip the pole-zero pattern to the desired zero-pole pattern is found to be: \( d_c = -0.1364 \). As the feed-through term \( d \) should be such that \( d < d_c \), the feed-through term selected for simulations is \( d = -0.137 \). The colocated pole-zero patterned magnitude response as well as the flipped zero-pole patterned magnitude response are shown in Figure 5(a). The damping controller gain is chosen by numerical search from Root-Locus plot given in Figure 5(b). This is \( C_d = \frac{1710}{2} \).

The open-loop and closed-loop magnitude response for \( G_{dy}(s) \) is shown in Fig. 6(a). Substantial damping
is noted for all three resonant modes. Fig. 6(b) shows the open-loop undamped and closed-loop damped step responses for the tip displacement. The step response settling times are measured to be 15.3s and 0.18s for the undamped and damped beam respectively. This clearly indicated the superior damping performance the Integral Resonant Control delivers and also demonstrates the effectiveness of the mathematical formulation of the adequate feed-through term derived in this work.

5 Conclusion

A rigorous mathematical proof is given to derive the adequate feed-through term needed to precisely flip the pole-zero interlacing pattern exhibited by the colocated systems into a zero-pole interlacing pattern. This then allows the accurate design of Integral Resonant Control scheme for multi-mode systems. With the full characterization of the effect of the feed-through term on the pole and zero location of the colocated system transfer function known, further inroads into optimizing the IRC scheme to deliver good damping over a number of performance metrics such as input disturbance rejection, sensitivity to sensor noise, control input requirement etc., can be investigated. This understanding of the effect of feed-through term could potentially allow similar simple yet high-performance damping schemes to be formulated for non-colocated systems.

References


Figure 6. (a) Magnitude response of the open-loop undamped (---) and closed-loop damped (—) transfer function $G_{dw}(s)$. (b) Step response of the open-loop undamped (---) and closed-loop damped (—) tip-displacement profile.


