A NEW INTERPRETATION OF ORDER-CHANGES IN TERMS OF MODE-SHIFTS IN A DESCRIPTOR SYSTEM

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ABSTRACT
A descriptor form that is convenient to investigate the effects of order-changes caused in a certain manner is proposed. Using this form, such effects are given new interpretations in terms of associated shifts of modes among the exponential, the static, and the impulsive modes of a descriptor system. It is shown that when the order is reduced, a dynamic mode changes to a static/impulsive mode that is disconnected to the output and is impulse-unobservable. Such a static/impulsive mode can be interpreted as a reminiscence of a dynamic mode that disappeared from but remained within the system expression after the order reduction. While this interpretation is obtained under a particular rule of order-changes, the basic insights obtained should be useful in analysing and designing control systems that can involve, directly or indirectly, order changes.

KEY WORDS
Modelling and Simulation, Order Changes, Exponential/Static/Impulsive Modes, Descriptor Form.

1. Introduction
Two systems which have different orders can have almost identical responses. To investigate if these systems are related in some ways, it is desirable to express them in a same form of models such that one can be obtained as a limit of the other. For instance, the order decreases when a parameter of the highest order term approaches zero or the order increases when the coefficient of a higher order term appears, in a transfer function. In this sense, a transfer-function expression can be a continuous one under order changes [1]. A point to note is that a transfer-function with monic denominator polynomial has a fixed order and order changes are not accommodated. Therefore, the denominator should be left as non-monic and the coefficient of the highest order term is made to approach zero or changed from zero to a finite value, so that the order can be reduced or increased. In contrast, such changes may be tedious to handle using the state-space form, because every time the order is changed, its dimension and usually parameters have to be modified. A change in the order is a discontinuous change in the state-space form [1]. Without knowing how the parameter values may be affected by the order change, its consequences in controller parameters are unclear; a change in a plant parameter might cause changes in all the controller parameters in discontinuous manners [1].

To overcome the issue, an integral-type state-space equation is proposed in [2], [3]. A generalized state-space equation is also proposed and has become widely known as a descriptor system [4] – [6]. Descriptor forms are used in such areas as large-scale systems [7], [8], singularly perturbed systems [9], [10], non-causal systems such as differentiators [11], switched systems [5], [12], and inverse systems [8], [13]. A descriptor system in general has three modes; the exponential, the static, and the impulsive, modes. Eigenvalues of the exponential mode are finite and have exponential responses. This mode is a dynamic mode which can be expressed in a state-space form. Eigenvalues of the static and the impulsive mode are infinite. The response of the static mode is the same as the input multiplied by a constant, which implies that it will show a jump if the input has a jump. The impulsive mode differentiates (may be multiple times) the input. These responses may remain inside the system or appear at the output, depending on the observability of the system [14] – [16].

As mentioned earlier, the denominator polynomial of a transfer function is often made to be monic, so that can be realized as a state-space expression. However, making the characteristic equation monic is essentially equivalent to declaring that the system order is fixed and an order change cannot be accommodated thereafter. In contrast, the descriptor form can provide an internal expression of systems and an explanation of the mechanics of various modes and interactions among them. Such versatility is used in the present study to relate order-changes to the static/impulsive modes of a descriptor system. This also gives an alternative view to the phenomena of large discrepancies in the responses at the initial time caused by an order reduction [10].

2. System Expressions
Several forms are known for the state-space equations as realizations of a monic transfer function [17]. In the fol-
lowing, three descriptor forms are considered as realizations of a transfer-function that undergoes order-changes. To this end, let the descriptor system be written as

\[ E\dot{x}(t) = Ax(t) + Bu(t), \quad Ex(0) = Ex_0 = 0, \]  

\[ y(t) = Cx(t), \]  

where matrices and variables are of compatible dimensions [5]. This form is capable of expressing a transfer function that is strictly-proper, bi-proper, or improper, as the case may be.

### 2.1 Parallel Form

Consider a system expressed in the \( n \)-th order transfer function of the following form:

\[ G_n(s) = \frac{b_n}{f_n(s)} = \frac{b_n}{a_n(s\rho_n + 1)(s\rho_{n-1} + 1)\cdots(s\rho_1 + 1)}, \]  

where \( a_n b_n \neq 0 \) and all the time-constant \( \rho_i \) are assumed to be distinct and nonzero initially. When one of them is set to be zero, the order is reduced by one and this order-change causes a continuous change in the characteristic equation. If the characteristic polynomial is made to be monic; that is, poles are explicitly shown as \((s+1/\rho_i)\), then they approach infinity and the expression fails, losing the continuity of system forms under the order-change.

A descriptor form of this transfer function can be written as in eqs. (1) and (2), where

\[ E = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \rho_n \end{bmatrix} \in \mathbb{R}^{n\times n}, \]  

\[ A = -I_n, \quad B = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T, \]  

\[ C = \frac{b_n}{a_n} \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}, \quad c_j = \frac{-\rho_j^2}{\prod_{i=1, i \neq j}^{n}(\rho_i - \rho_j)} \]  

It is to be noted that the arguments to follow are applicable also to a transfer function with a numerator of up to \((n-1)\)th degrees, with obvious modifications to parameters in \( C \). When the transfer function is bi-proper or improper, the results are applicable to the strictly proper portion of the system and the dimension of the descriptor vector must be expanded accordingly.

[Order Reduction Procedure 1]

If the order of this system is to be reduced by one, it is probable that the fastest pole is ignored. For instance, if \( \rho_n \) is the smallest value (fastest time-constant), this can be set to zero to let the corresponding pole go to infinity and the order of the system reduce by one. This implies that the transfer function has a continuous change under the order change, as

\[ G_{n-1}(s) = \frac{b_n}{a_n(s\rho_n + 1)(s\rho_{n-1} + 1)\cdots(s\rho_1 + 1)}, \]  

To see how the poles of \( f_n(s) \) approaches those of \( f_{n-1}(s) \) as the coefficient \( \rho_n \) approaches zero, write \( f_n(s) \) as

\[ f_n(s) = (s\rho_n + 1)a_n(s\rho_{n-1} + 1)\cdots(s\rho_1 + 1) = (s\rho_n + 1)f_{n-1}(s). \]  

This shows that as \( \rho_n \) is changed, only the pole at \(-1/\rho_n \) is affected and all other poles unaffected. When increasing the order by one without changing pole locations, just add a term with a large time constant. The corresponding descriptor form, eqs. (4) – (6), will also have no difficulty in setting \( \rho_n = 0 \), while the ordinary state-space equation will.

### 2.2 Kronecker Form

A descriptor form similar to the above is the Kronecker form, where the poles are arranged such that \( \rho_n < \rho_i \) for any \( i \) and divided into the slow and the fast sets of variables as conveniently chosen. The system can then be transformed into the following Kronecker form [6]:

\[ E = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, A = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}, B = \begin{bmatrix} B_S \\ B_F \end{bmatrix}, C = \begin{bmatrix} C_S & C_F \end{bmatrix}. \]  

where matrices have compatible dimensions. The slow variables form \( x_s \) and the fast ones \( x_F \) so that the descriptor vector is given by

\[ x = \begin{bmatrix} x_s \\ x_F \end{bmatrix}. \]  

When all \( \rho_i \) are nonzero, \( E \) can be inverted and the form is equivalent to a state-space equation. When \( \rho_n = 0 \) and all others are nonzero, then \( N = 0 \), which corresponds to one given by eqs. (4) and (5). This slow-fast decomposition is used in [18] to propose a mapping discrete-time model and to show the structure of this decomposition. Figure 1 shows this structure, which is valid even when \( N \) approaches the zero matrix; i.e., the closed-loop in the fast mode component \( x_F \) opens at this limit but still a valid expression. Thus, the system expression is continuous under order changes.

![Figure 1. Structure of the Kronecker Form in a Continuous Expression][18]
2.3 Phase Variable Form

To consider order-changes, besides eq. (3), the following form is also possible:

\[ G_n(s) = \frac{b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}, \]

(11)

where \( b_0 \neq 0 \) and \( a_i \neq 0 \) for all \( i \) initially. Let the \( n \)th-order characteristic polynomial be written as

\[ f_n(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0. \]

(12)

To see how the poles of \( f_n(s) \) approaches those of \( f_1(s) \) as the coefficient \( a_n \) approaches zero, rewrite the above as

\[ \frac{f_n(s)}{f_1(s)} = 1 + \frac{1}{a_n} \cdot \frac{f_{n-1}(s)}{s^n} = 0. \]

(13)

This shows that all poles are affected in general as \( a_n \) is changed. When \( a_n \) large, all the poles are clustered around the origin. However, our interest is on the poles when \( a_n \) is zero; they converge to those poles of the lower order system. What’s more, when the change in the coefficient of the highest order term is small, its effects on the pole locations is also small.

[Order Reduction Procedure 2]

When decreasing the order by one, only the highest order term may be ignored if its coefficient is sufficiently small. When increasing the order by one without changing the pole locations too much, a term of a higher order with a sufficiently small coefficient may be added, while other coefficients are kept intact. This will introduce a fast (but finite) pole.

The realization of the transfer function as the descriptor-form that is proposed in this paper, and used in the discussions below, is the following: The system whose transfer function is given by eq. (11) can be written as eqs. (1) and (2), where

\[
E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & a_n \end{bmatrix} \in \mathbb{R}^{n 	imes n} \quad \text{(14)}
\]

\[
A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \in \mathbb{R}^{n 	imes n} \quad \text{(15)}
\]

\[
B = \begin{bmatrix} b_0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{n+1}, \quad C = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{1 \times n} \quad \text{(16)}
\]

When \( a_0 \neq 0 \), eqs. (14) – (16) can be converted into the standard state-space equation, by multiplying both sides by the inverse of matrix \( E \), which is non-singular. When \( a_n = 0 \), the equation in last row becomes the constraint among the variables. Any further coefficient \( a_i \) becoming zero requires no change in the form. These changes in the coefficients are further considered in the next section.

3. Interpretation of Order Changes

Based on the phase-variable form, a new interpretation of order changes is explained below.

3.1 Problem Statement

The problem to be solved is the following: For the system given by eq. (11), reduce the order one by one, by setting the coefficient of then the highest order term to zero successively.

3.2 Dimensions of The Three Modes

The descriptor system eq. (1) is solvable uniquely if its pencil is regular; that is if \( \text{det}(sE - A) \) is not permanently zero, which holds if \( a_i \neq 0 \) for all \( i \). The number of exponential modes is given by

\[ d = \deg(\text{det}(sE - A)). \]

(17)

For the system with \( n \) descriptor variables, the state-space equation can express \( d \) exponential modes, leaving \( n-d \) modes uncovered. The static and impulsive modes make up this difference. The number of variables associated with the static modes is given by

\[ r = \text{rank}(E). \]

(19)

The number of variables in the impulsive modes is given, therefore, by

\[ r - d. \]

(20)

Thus, \( r \) equals the sum of the numbers of exponential and impulsive modes. The sum of all three modes constitutes the overall system, whose dimension is \( n \). Figure 2 illustrates these relationships.

![Figure 2. The Three Modes and Their Dimensions.](image-url)
Starting from the full order \( n \), chosen based on a certain criterion, let the coefficient \( a_n \) of the highest order to be zero, and then \( a_{n+1} = 0 \), and so on, in eqs. (14) – (16). Taking into account the numbers of variables in each mode, five cases of the order changes are identified as listed in Table 1. After the first \( i \) leading coefficients are zeroed, eqs. (14) – (16) can be broken into each component and then regrouped into the three modes according to cases in Table 1, as follows:

<table>
<thead>
<tr>
<th>Case</th>
<th>( a_i )</th>
<th>( r )</th>
<th>( d )</th>
<th>Exp. Mode</th>
<th>Static Mode</th>
<th>Imp. Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( a_0 \neq 0 )</td>
<td>( n )</td>
<td>( n )</td>
<td>( n )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>( a_0 = 0 ), ( a_{i+1} \neq 0 )</td>
<td>( n-1 )</td>
<td>( n-1 )</td>
<td>( n-1 )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>( a_0 = a_{i+1} = 0 ), ( a_{i+2} \neq 0 )</td>
<td>( n-1 )</td>
<td>( n-2 )</td>
<td>( n-2 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>( a_0 = a_{i+1} = a_{i+2} = 0 ), ( a_{i+3} \neq 0 )</td>
<td>( n-1 )</td>
<td>( n-3 )</td>
<td>( n-3 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>( a_0 \sim a_{i+1} = 0 ), ( a_{i+2} \neq 0 )</td>
<td>( n-1 )</td>
<td>( n-i )</td>
<td>( n-i )</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The first of the three is the exponential portion of the system, which has the dimension of \( d=n-i \) and can be arranged into the \( n-i \)-dimensional state-space equation as

\[
\begin{bmatrix}
\dot{x}_1 \\
\vdots \\
\dot{x}_{n-i}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & \ddots & 0 & 1 \\
0 & \cdots & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\vdots \\
x_{n-i}
\end{bmatrix} +
\begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix} x_{n-i+1},
\] (21)

which is in the controllable-canonical-form from the input \( x_{n-i+1} \) to the output \( x_1 \), and forms the \( (n-i) \)-th order integrators.

The second comes from the last row of eqs. (14) – (16), and is the static mode with only one variable that is determined by the state variables determined in the exponential mode. It is given by the algebraic constraints as

\[
a_{n+1}x_{n-i+1} = \sum_{j=1}^{i} a_{n+1-j} x_{n-i+1-j} + u,
\] (22)

whose input is \( u \) when \( a_n = 0 \). It should be noted that this is always one dimensional and given by \( (n-i+1) \)-th variable, which acts as the input to both the exponential mode and the impulsive mode, which is explained next. When \( a_n \neq 0 \), there is no static and impulsive modes and \( u \) goes straight into the exponential mode. The last row of the controllable canonical form acts as the static mode.

The third is the impulsive mode given in a descriptor format as

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
1 & 0 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_{n-i+1} \\
\vdots \\
x_{n-2} \\
x_{n-1}
\end{bmatrix}
= \begin{bmatrix}
\cdots \\
\cdots \\
x_{n-i+1} \\
\cdots
\end{bmatrix} +
\begin{bmatrix}
-1 \\
0 \\
\vdots \\
0
\end{bmatrix} x_{n-i+1},
\] (23)

where the input is again \( x_{n-i+1} \), as in the exponential mode. It may be noted that the dimension of the impulsive mode is \( i \), which is one larger than the number of variables actually needed; the top row of eq. (23) is used to form a convenient expression but not used in the impulsive mode.

From these three equations, and the output equation eq. (2) with eq. (16), the block diagram shown in Figure 3 can be drawn.

Figure 3 illustrates that the exponential mode is a series of integrators, while the impulsive mode a series of differentiators. The static mode determines the gains of state feedback from the exponential mode, disposing the former states, which are now impulsive mode variables, from the feedback. Eqs. (21) – (23) are equivalent to eqs. (14) – (16),
but have different dimensions so that these two set of equations are not related by transformations.

When the next highest order coefficient is already zero in reducing the order by one, then the order may be reduced further till the next, nonzero, highest-order coefficient is found.

3.3 Impulse Observability

For any descriptor system expressible in eqs. (1) and (2), there always exist two non-singular matrices $Q$ and $P$ such that [6] these are transformed into the non-unique equivalent descriptor form given by

$$E^s = QEP = \begin{bmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad (24)$$

$$A^s = QAP = \begin{bmatrix} A_{11}^s & A_{12}^s \\ A_{21}^s & A_{22}^s \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad (25)$$

$$B^s = QB = \begin{bmatrix} B_1^s \\ B_2^s \end{bmatrix} \in \mathbb{R}^{n \times p}, \quad (26)$$

$$C^s = CP = \begin{bmatrix} C_1^s \\ C_2^s \end{bmatrix} \in \mathbb{R}^{m \times n}. \quad (27)$$

A descriptor system is said to be impulse observable if knowledge of $y(r)$ is sufficient to determine $x(r)$ for every real $r$ [14]. It is known [15], [16] that the descriptor system given by the above is impulse observable if and only if the following condition holds:

$$\text{rank} \left[ \begin{bmatrix} A_{12}^s \\ A_{22}^s \end{bmatrix} \right] = n - r. \quad (28)$$

Since the system given by eqs. (14) – (16) is in this form, this condition implies that

$$\text{rank} \left( \begin{bmatrix} a_{n-1} \\ 0 \end{bmatrix} \right) = \begin{cases} 1 & (a_{n-1} \neq 0) \\ 0 & (a_{n-1} = 0) \end{cases}. \quad (29)$$

Therefore, all the static and the impulsive modes given above are impulse unobservable; i.e., there will be no step or no impulse component in the output unless there are such components and their derivatives in the input. They may appear inside the descriptor form as reminiscences of exponential mode that existed before the order reduction. In a different context, discrepancies between the initial condition of a system and that of its reduced order systems have been known and explained as an effect of a boundary layer [10], [11]. An alternative explanation of such differences may be the above mechanism of static/impulsive mode generation due to an order change as a cause of discontinuities.

4. Application to Control

Controller design may be considered as an act of adjusting coefficients of a characteristic polynomial, where the design specification is given as a set of desired coefficients. In general, for a characteristic polynomial in an anti-monic form, the sequence of its coefficients follows a certain pattern up to a certain order and tends, for stability, to approach zero as the order increases beyond [19]. In [1], partial model matching is proposed, where the coefficients of low order terms in the characteristic equation are altered, while those of higher order terms, which approach zero and do not affect the shape of a step response very much, are ignored. In view of Figure 3, increasing the coefficients of the lower order terms in effect makes the higher order terms relatively small, implying the indirect order reduction.

Since the change of signs of the highest order coefficient affects the stability/instability of the system, making this coefficient small to reduce the order directly, should be done carefully, however. For instance, a safety measure should be placed in forcing the leading coefficient of the characteristic polynomial approach zero from a safe side; i.e., at least preventing the leading coefficient from changing its signs by order reduction.

5. Conclusion

It has been shown using the proposed descriptor form that an order change could be considered as a shift of a variable from an exponential mode to a static/impulsive mode, and vice versa. The static and the impulsive modes, which are the modes that cannot be handled by the state-space expression, can be interpreted as a piece of reminiscent information of an exponential state, which remains in the descriptor form but disconnected from the output due to the order reduction.

While the rule of order changes is a particular one, this gives a convenient visualization in terms of simple variable-shift among the three modes. Since there are many other order-reduction methods [20] and they may lead to other insights into the subject, they should be investigated.

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References


