MODEL ORDERS AND NUMERICAL ISSUES IN
LLC RESONANT DC-DC CONVERTERS

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ABSTRACT

The condition number of a Sylvester matrix composed of
the denominator and numerator polynomial of a model is
investigated numerically for an LLC resonant dc-dc
converter. The order of the model is changed from six to
one in a certain manner and they are discretized using the
step-invariant and matched-pole-zero methods expressed
in the delta and shift operators. Their condition numbers
show that in both operator forms, the higher the order is,
the larger the condition number becomes, implying poorer
numerical conditioning. It was also found that the step-
invariant model is more ill-conditioned than the matched-
pole-zero models in general. While the number becomes
constant as the sampling frequency increases for the delta
cases, it keeps increasing for the shift cases, which is well
known. However, in the range of sampling frequencies
expected in recent practices, the models expressed in shift-
form have smaller condition numbers and are better-
conditioned than those in delta-form of respective orders.

KEY WORDS

Power and Energy Systems, LLC Resonant Converter,
Condition Number, Sylvester’s Matrix, Plant Order,
Discretization.

1. Introduction

Model-based designs of various control systems require
simple and yet sufficiently detailed plant models. This
requires the determination of a finite plant-order, which is
in fact infinite. Such systems with a finite number of inputs
and outputs would translate into uncontrollable and
unobservable models, which call for a compromise
between simplicity and complexity. In an effort to design a
digital controller for regulating the output voltage of an
LLC converter under dynamic load transients [1-2], it is
reported in [3] that plant models derived from experimental
frequency responses can be exceptionally accurate if the
order is chosen to be six. However, this model is
numerically ill conditioned so that a model-based digital
controller design cannot be completed; a Matlab program
stops due to a large condition number of Sylvester matrix
of the discrete-time plant model expressed in the delta
operator. It also reports that if the plant model is reduced to
second-order, the controller design can be completed.

Numerical difficulties are encountered in many fields
of mathematics and engineering, including solutions for
algebraic equation [4], parameter identification [5],
singular perturbation [6], and controller designs [7]. A
condition number is often used to assess the degree of
numerical difficulty. For instance, correlation matrix is
used in [5], which concludes that the condition number
becomes large when (i) the levels of the input and the
output signals differ significantly, (ii) there is a pole
located closely to the unit circle in the z-domain, and (iii) a
pole and a zero are located closely together. In a
polynomial design of a control system based on state-feedback
with a state-observer, Sylvester’s matrix is used
to solve a Diophantine equation [7]. The Sylvester matrix
can be used to see if two polynomials are relatively prime
(coprime) [8]. When this is applied to the denominator and
the numerator of a transfer function, the joint
controllability and observability can be determined [9]. In
[7], numerical difficulties are investigated based on
the condition number of the Sylvester matrix of continuous-
time and discrete-time models. It concludes that the
difficulty in the z-form can significantly be alleviated using
the delta-form. Discrete-time models expressed in the usual
shift or z operators can have significant numerical
disadvantages and in the condition number of plant’s
Sylvester matrix. This is in accordance with the statement
given in [10] that the delta and Euler operators have
superior numerical properties than the q and z operators at
high sampling frequencies. However, the present paper
points out that the technology may not be able to realize
sufficiently high sampling frequency to realize this, such as
those used for modern LLC converters.

The step-invariant-model (SIM) [11] of the plant is a
discrete-time model that represents a combination of the
zero-order-hold, the plant, and the sampler. It is known that
for a stair-case type input (a series of step changes) with
the period equaling the sampling interval, the SIM gives
the response that is identical to the response of the
continuous-time plant at the sampling instants. However,
while all the poles of a SIM have their corresponding
continuous-time counterparts, not all the zeros do and some
are extraneous ones [11]; it introduces the so-called
sampling zeros, which approach finite values as the
sampling period T goes to zero. The SIM usually yields
some extra finite zeros due to discretization [11]. For the shift form case, these zeros are the roots of the Frobenius polynomial, which appear as $T \rightarrow 0$, while they approach infinity for the delta form case. They are called the sampling zeros and may interfere with poles causing large condition numbers. The matched-pole-zero (MPZ) models discretize the poles in the same manner as the SIMs do. Moreover, the finite zeros are mapped according to the same rule as the poles, introducing no sampling zeros. In the following, both the SIM and MPZ discrete-time models expressed in the shift and the delta operator forms will be used. Since analyses of such phenomena are intractable, simulation studies are carried out.

2. Models of LLC Resonant Converter

2.1 Continuous-Time Model

The plant under consideration is an LLC resonant dc-dc converter MMX904-FLC-DEMO(Digital) (Global Micronics Inc.) shown in Figure 1. [3], whose input is a change in the switching frequency $f_{sw}$ with nominal frequency of about 110kHz and the output is a constant voltage of 12V. Since the input is not physically measurable, the excitation input is injected in the feedback path and the output voltage of the converter is observed using the frequency response analyzer FRA 5095 (NF Electronic Instruments). The plant transfer function was then calculated from the model of this experimental frequency response data [3]. The one used in the present study was obtained from the experiment where there were 150W (12A) loading under the integral control $C(s)$=5500/s. To fit the model accurately to the data, and taking the Nyquist frequency into account, the order of the plant model was chosen to be six. The transfer function of the LLC converter was obtained [3] as

$$\tilde{G}_o(s) = \frac{0.3591}{1 + 1.103 \times 10^{-7} s + 6.266 \times 10^{-8} s^2 + 2.396 \times 10^{-11} s^3 + 6.344 \times 10^{-14} s^4 + 9.969 \times 10^{-23} s^5 + 6.675 \times 10^{-38} s^6}$$

In the present study, the characteristic polynomial is normalized such that the coefficient of the lowest order term becomes unity. It can be seen that the magnitude of coefficients progressively decreases as the order increases.

2.2 Order Changes

Let the $n$th-order characteristic polynomial be written as

$$f_n(s) = 1 + \alpha_n s + \alpha_n s^2 + \ldots + \alpha_n s^n.$$  

(2)

To see how the poles of $f_n(s)$ moves as the coefficient $\alpha_n$ changes from zero to positive infinity, rewrite the above as

$$\frac{f_n(s)}{f_{n+1}(s)} = 1 + \frac{\alpha_n s^n}{f_{n+1}(s)} = 0.$$  

(3)

This shows that all the poles of $f_n(s)$ of $0$ approach the origin as $\alpha_n$ increases, while they approach the poles of $f_{n+1}(s)=0$ as $\alpha_n$ approaches zero at which time the order is reduced by one. Such continuity is ideal in studying the effect of order changes, which causes continuous changes in the output.

Figures 2 and 3 show examples of how the $n$-th order plant poles move as $\alpha_{n+1}$ changes from 0 to infinity. When increasing the order by one without significantly changing pole locations, simply add a term of a higher order with a sufficiently small coefficient, which will introduce a fast but finite pole. When decreasing the order by one, the highest order term may be ignored if its coefficient is sufficiently small. If it is not small, then it suggests that the order should not be reduced further. Since the LLC converter model given by eq. (1) has coefficients that conform to this point of view, it was decided that only the lowest order coefficient is ignored and other coefficients are kept unchanged when the order is changed.

![Figure 2. Root-Locus Relating 5th and 6th Order Poles.](image-url)
The same principle is adopted in specifying the desired characteristic polynomials having similar dynamics with different orders [12]-[13]. The fact that coefficients of higher order terms do not affect the step responses also supports this principle [13]. While this is not applicable to just any systems, for those with successively decreasing coefficient as the order increases, as in the present case, this is a simple and good starting point. Therefore, in the following, the \((n-1)\)th order model is obtained by ignoring the coefficients of the \(n\)th degree term in \(s\); i.e., for \(n\) less than seven,

\[
G_{n-1}(s) = \frac{b_0}{1+a_1s+a_2s^2+\ldots+a_{n-1}s^{n-1}}.
\]

Figure 4 shows the Bode plots of plant models \(G_1\) to \(G_6\). Up to a few kHz, all the models look similar, and as the order increases, the plots matches those of the sixth order model up to higher and higher frequencies. Step responses are given in Figure 5, which shows that they are underdamped but as the order decreases, they become less oscillatory. Figure 6 shows the response of the plants for a step change in its reference voltage under integral control \(C(s) = 5500/s\). It can be seen by comparing Figures 4 and 6 that integral control is robust to variations of an order in the plant and gives more or less the same closed-loop responses, indicating that for the modest performance of output regulations, the first or second order models may be sufficient.

### 3. Condition Number of Sylvester Matrix

A system with a transfer function with coprime numerator and denominator is controllable and observable [9]. The condition number of the Sylvester matrix is often used to indicate a degree of non-coprimeness of two polynomials [8]. Since this yields long formula for the sixth order plant and the like, and the procedures are basically the same for any orders but tedious, only those for some low order cases are illustrated below.

#### 3.1 First-Order Systems

Consider the system with one real pole and one real zero expressed as

\[
G(s) = \frac{b_0 + b_1s}{a_0 + a_1s},
\]

where \(a_1 \neq 0\). \(s\) can be considered as the Laplace, Euler, or any other operators. Its Sylvester matrix is given by

\[
S = \begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix}.
\]

The condition number \(\kappa(S)\) is given, using 1-norm, by

\[
\kappa(S) = \max(|a_0| + |b_0|, |a_1| + |b_1|) \cdot \frac{\max(|a_0| + |b_0|, |a_1| + |b_1|)}{|\det(S)|}.
\]

\[
\det(S) = a_0b_1 - a_1b_0.
\]

Thus, the condition number approaches infinity as \(\det(S)\) approaches 0. When \(b_1 \neq 0\), rewriting eq. (5) as
\( G(s) = \frac{b_0 + s}{a_0 + s} = b_0 \beta + s \),
\[ \text{eq. (8)} \]
which gives
\[ \det(S) = (\alpha - \beta) a b, \]
indicating that when \( \det(S) = 0 \), there is a pole-zero cancellation in the plant transfer function, implying loss of joint controllability and observability [9]. This occurs when \( \alpha = \beta \), at which case the condition number is infinite.

An important special case is a pure integrator given by
\[ G(s) = \frac{b_0}{s}, \]
for which the condition number of Sylvester's matrix is
\[ \kappa(S) = \left[ \frac{|b_0|}{|b_0|} \right], \quad |b_0| \geq 1. \]

This shows that the condition number of the pure integrator with (positive or negative) unity gain has the smallest possible value, while it approaches infinity as \( b_0 \to 0 \).

### 3.2 Second-Order Systems

Let the second-order model be given by
\[ G(s) = \frac{b_0 + b_2 s + b_0 s^2}{a_0 + a_1 s + a_2 s^2}, \]
\[ \text{where } a_2 \neq 0. \]

Its Sylvester matrix can be written as
\[ S = \begin{bmatrix} a_0 & a_1 & a_2 & 0 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ 0 & b_0 & b_1 & b_2 \end{bmatrix}. \]

Its condition number is given by
\[ \kappa(S) = \frac{\max(f_1, f_2, f_3, f_4) \cdot \max(g_1, g_2, g_3, g_4)}{\det(S)}, \]

where
\[ f_1 = [a_0] + [b_0], \quad f_2 = [a_0] + [a_0] + [b_0] + [b_1], \]
\[ f_3 = [a_0] + [a_0] + [b_1] + [b_2], \quad f_4 = [a_2] + [b_2], \]
\[ g_1 = [b_0](a_0 b_0 - a_2 b_0) + [b_0](a_0 b_0 - a_0 b_2) + [b_2](a_0 b_0 - a_2 b_0), \]
\[ g_2 = [b_2](a_0 b_0 - a_2 b_0) + [b_2](a_2 b_0 - a_0 b_2), \]
\[ g_3 = [a_0](a_0 b_2 - a_2 b_2) + [a_0](a_0 b_2 - a_2 b_0), \]
\[ g_4 = [a_0](a_0 b_2 - a_2 b_0) + [a_0](a_0 b_2 - a_0 b_0) \]
\[ + [a_0](a_0 b_2 - a_2 b_0) + [a_0](a_0 b_2 - a_0 b_0). \]

When \( b_2 \neq 0 \), \( \det(S) \) can be written as
\[ \det(S) = a_0^2 b_0^2 - a_0 b_0 b_1 - 2a_0 a_2 b_1 + a_0^2 b_2^2, \]
\[ = (a_1 - \beta_1)(a_1 - \beta_2)(a_1 - \beta_3)(a_1 - \beta_4) a_0^2 b_2^2 \]

where
\[ \alpha_1 = -a_0 \sqrt{4a_0^2 - 4a_0 a_2}, \quad \beta_1 = -b_1 \sqrt{b_1^2 - 4b_1 b_2}. \]

Therefore, \( \det(S) \) is zero only when the numerator and denominator polynomials have at least one common factor.

### 3.3 Triple-Pole Systems

General third-order systems yield a long equation for the condition number. Therefore, only the following simple case is considered:
\[ G(s) = \frac{1}{(a_0 + s)^3} \]

Its condition number is given by
\[ \kappa(S) = \max(f_1, f_2, f_3, f_4) \cdot \max(g_1, g_2, g_3, g_4) \]

where
\[ p_1 = [a_0] + 1, \quad p_2 = [a_0] + 3[a_0] + 3[a_0] + 1, \]
\[ q_1 = [a_0] + 3[a_0] + 3[a_0] + 1, \quad q_2 = [a_0] + 6[a_0] + 3[a_0] + 3[a_0] + 1. \]

When \( a_0 \) is zero, the system is a triple integrator and its condition number is the minimum of unity.

The condition numbers of the continuous-time (CT) LLC converter plant are shown in Table 1. It is seen that, compared with the reciprocal of the machine accuracy \( \varepsilon^{-1} = 4.5036 \times 10^{15} \) (Matlab R2016b), they are rather large for any order, especially for orders larger than three, and get larger as the order increases. While the separation of pole locations may partly be to blame, it is suspected that the fast dynamics (small time-constant) is a more prominent factor for ill-conditioning, as seen in the first-order case with a small \( \det(S) \) due to small \( a_0 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \kappa(S) )</td>
<td>3.1112 \times 10^{16}</td>
<td>1.0634 \times 10^{13}</td>
<td>3.5323 \times 10^{10}</td>
<td>1.9976 \times 10^{7}</td>
<td>4.9259 \times 10^{3}</td>
<td>1.5255 \times 10^{2}</td>
</tr>
</tbody>
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### 4. Discrete-Time Models

#### 4.1 Step-Invariant and Matched-Pole-Zero Models in Delta Form

Since analyses of how the condition number of the Sylvester matrix is affected by the discretization method and the form of expression, are difficult, simulation studies are carried out. Figure 7 shows the results obtained using the SIM models in the delta-operator. The order of the SIM is decreased one at a time from six to one, by simply ignoring the coefficients of the highest order term, as in the
s operator case for the continuous-time domain. It can be seen that the condition number increases as the order does in general. For the sampling frequencies higher than the range of switching frequency $f_{sw}$ used in practice, which is shown by a pair of vertical lines in the figure, the condition numbers become constant approaching those of the continuous-time (CT) model. Below the switching frequency, the SIM generally give condition numbers that are sensitive to the sampling frequency, especially with the third and fourth order cases halting computations in Matlab due to inf warnings at times.

Figure 8 shows the condition numbers of the matched-pole-zero (MPZ) models, which are the discrete-time models whose zeros as well as poles are discretized in the same manner as the poles of the SIM. In general, MPZ models have smaller condition numbers and smoother changes than the SIM in the lower-half range of the sampling frequencies tested. Moreover, these numbers are even smaller than those of the continuous-time cases. It is suspected, therefore, that the sampling zeros that appear in the SIM have something to do with the ill-conditioning.

4.2 Step-Invariant and Matched-Pole-Zero Models in Shift Form

It is known [7,10] that, in general, the shift or $z$ operator form is numerically inferior to the delta or Euler operator form as the sampling frequency approaches infinity, while the opposite may be the case for smaller frequencies [10]. Simulation results shown in Figures 9 and 10, respectively, are for the shift-form SIM and MPZ models, which are obtained, respectively, from the delta-form SIM and MPZ models of the same orders. They suggest that in the range of frequencies tested, the condition numbers are considerably smaller with the shift-operator forms than those with the continuous-time model, and significantly smaller than the delta-operator SIM and MPZ models for any plant orders. While it is known that the shift-form model becomes progressively more ill-conditioned as the sampling frequency increases and the required number of bits to retain numerical accuracy goes to infinity [10], this happens only beyond around 10MHz for MPZ and well beyond that for SIM.

5. Conclusion

Through extensive simulations on the condition numbers of the Sylvester matrix for the LLC resonant dc-dc converter used in [3], the following observations have been made:

- The condition numbers of the continuous-time models of any order have rather large values; implying ill-conditioning for numerical computations. It becomes larger as the order increases.
- The delta-form models (both SIM and MPZ) had the condition numbers that are close to those of the continuous-time models in the range of common sampling frequencies, while the shift-form had varying condition numbers in all frequencies.
- The SIM and the MPZ models of all orders had smaller condition numbers in shift-form than in delta-form, in
the range of sampling frequencies expected in practice. This is contrary to our previous findings.

- It is suspected that a small time constant (fast dynamics) is a major cause of ill-conditioning for this particular plant. It is also suspected that the sampling zeros in the SIM have something to do with this issue and require further study.

Considering the range of sampling frequencies expected in recent practices,

- if the delta-form is used, either SIM or MPZ model could be used, since they both have the condition numbers that are the same as those of the continuous-time system; i.e., they are as good as the continuous-time model numerically.
- if the shift-form is used, the MPZ models have smaller condition numbers, especially in the lower end of the switching frequency range.

As faster switching regulators become available, dynamics of LLC converters are expected to become faster and, thus, the issue of numerical ill-conditioning seems to be getting more prominent. While the delta-operator form has been referred to being better-conditioned numerically than the shift-operator form in many cases, it should be cautioned that this is not always the case, depending on the plant under consideration and the sampling frequency to be used.

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References