A CRAMÉR RAO BOUND APPROACH FOR EVALUATING THE QUALITY OF EXPERIMENTAL SETUPS IN ELECTRICAL IMPEDANCE TOMOGRAPHY

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ABSTRACT
In this report, we propose the application of the Cramér Rao Lower Bound (CRLB) as a performance measure for optimal design of experimental setups in electrical impedance tomography. In particular, we focus on the optimum positioning of electrodes. Cramér Rao Bound is bounded from below by the inverse of the Fisher information matrix (FIM). FIM incorporates all aspects of the forward problem, statistical properties of the measurement noise, and multi-frequency data. We consider the application of CRB in both the deterministic as well as the Bayesian setting. We first present the CRB for the case of the unbiased estimator and then the Bayesian Cramér Rao Bound (BCRB) for the case of the biased estimator. All CRB computations are performed using a measured noise model from a clinical experiment.

KEY WORDS
Electrical Impedance Tomography, Cramér Rao Bound, Fisher information matrix

1 Introduction
The objective in electrical impedance tomography is to estimate the unknown electromagnetic properties, namely the permittivity and conductivity in a given domain from a finite number of voltage measurements made on it’s boundary [1, 2]. It is a parameter estimation problem, and the subject of much interest in inverse problems and signal processing [3, 4]. Furthermore, EIT has numerous medical and industrial applications [5–16].

The mapping between the unobservable parameters and the data is nonlinear [17]. This mapping, is often referred to as the forward problem [18], or according to signal processing literature, the mathematical model. In general a setting, the estimation problem reduces to a large-scale nonlinear optimization problem [19–22]. This approach can be employed in process tomography, where the mismatch between model and the experimental setup can be minimized. However, in many medical applications, an accurate modeling of the experimental setup is a challenging step. Various sources of error can arise as a result of inaccurate positioning of the electrodes, boundary shape, and the unknown conductivity of the background. In low contrast applications, one can employ a linear approach to image reconstruction. A well known linear method is time-difference imaging [20, 23]. There are two main assumptions involved in time-difference imaging. The first assumption is that constitutive parameters vary as a function of time in animal, human subjects or some physiological process. The second assumption is that the variations in the conductivity are small about some background value. It is therefore a suitable approach for imaging of low contrast objects. Bayesian linear and nonlinear filtering techniques is an active area of research in EIT [20, 21, 23–30]. The estimation problem in EIT is an ill-posed problem [3, 18, 31]. One must employ some regularization strategy in order to obtain a reasonable estimate [18]. In practice, the most common approach is to recast the estimation problem as a Bayesian inverse problem [4, 18, 32]. In Bayesian statistics all quantities are modeled as random variables. It is well known that the accuracy of Bayesian methods rely on a good knowledge of the probability density function of the parameters (priors) [4, 18], which may not always be available. In microwave tomography a robust regularization strategy based on the concept of the Fisher information matrix (FIM) [4] has been proposed by [33–35]. This regularization approach has been coined as the Fisher based preconditioning (FBP) method. Recently the method has been extended to EIT [16, 22]. In the same study, the authors presented nonlinear image reconstructions of two high contrast cylindrical targets using multi-frequency experimental data. The FBP scheme involving an iterative quasi-Newton type algorithm (BFGS) [19] proved to be successful in the imaging of the targets. In linear approaches to image reconstruction, a one-step using the FBP approach has similarities to the "Newton’s One-Step Error Reconstructor" (NOER) algorithm [36]. The original NOER algorithm did not account for statistical properties of the measurement noise. However, in a recent study, the measurement noise was introduced in the estimation model and a nonlinear regularization strategy was employed to regularize the
In this study, we propose the Cramér Rao bound (CRB) as the metric for assessing the quality of experimental setups in electrical impedance tomography. CRB is a standard tool in signal processing and system identification [4, 38]. Furthermore, it is often used to assess the quality of a system even before it is built. It allows for bench marking and comparison of the performance of unbiased estimators. CRB is bounded from below by the inverse of the FIM. All information about the forward problem, statistical properties of the measurement noise, and multifrequency data [4, 22, 38] is encoded in the FIM. This makes CRB, an ideal metric for assessing the quality of experimental setups. It needs to be mentioned CRB has been extended to Bayesian approaches to parameter estimation and nonlinear filtering [39]. The outline of the paper is as follows: In section 2, a brief overview of EIT is presented. We then describe a linearized approach to image reconstruction in section 3. In section 3.1, the CRB is presented for the case of deterministic parameters. The Bayesian Cramér Rao bound (BCRB) is presented in sections 3.2. In section 4, CRB computations are presented for different test cases. Finally, the conclusion is presented in section 5.

2 Formulation of the Problem

Electrical impedance tomography needs to be formulated as a nonlinear filtering problem [39, 40]. To define the problem of nonlinear filtering in EIT, we consider the conductivity as the target state vector $\sigma[k] \in \mathbb{R}^N$. Where N is the dimension of the state vector and $k \in \mathbb{N}$ denotes the time index. The conductivity evolves according to the following discrete-time stochastic model,

$$\sigma[k] = f_{k-1}(\sigma[k-1], n[k-1])$$  

where $f_{k-1}$ is a known possibly nonlinear function of the conductivity $\sigma[k-1]$ and $n[k-1]$ is referred to as the process noise sequence. The process noise accounts for modeling errors in the state evolution model. Here, the objective is to recursively estimate the conductivity $\sigma[k]$ from the voltage measurements $v[k] \in \mathbb{R}^M$. Where, $M$ denotes the dimension of the of measurements vector. The measurements are related to the target state via the measurement equation:

$$v[k] = h_k(\sigma[k], w[k])$$

The function $h : \sigma \mapsto v$ is a known nonlinear mapping [17] and $w[k]$ is a measurement noise sequence. The voltages are uniquely determined from the solution of the partial differential with boundary conditions based on the complete electrode model [17]. Here, the PDE is solved using the finite element method [41, 42].

3 Time-Difference Imaging

Time-difference imaging requires the global linearization of equations (1) and (2). The state and measurement equations read as

$$\delta \sigma[k] = F_{k-1} \delta \sigma[k-1] + n[k-1]$$  

(3)

$$\delta v[k] = J_k \delta \sigma[k] + \tilde{v}[k]$$

(4)

Where $F_{k-1} \in \mathbb{R}^{N \times N}$ and $J_k \in \mathbb{R}^{M \times N}$ respectively denote the Jacobian matrix of state evolution and the observation model. Furthermore, $\delta v[k]$ and $\delta \sigma[k]$ denote the perturbations in voltage and conductivity respectively. It follows from equation (2) that the number of the rows of $J_k$, is a function of the number of electrodes in the experimental setup. It is expected, that as more electrodes are added (M increases), then so will the estimation accuracy. Incorporating multi-frequency data, also features as an increase in $M$ [22]. Here we compute $J_k$ using the adjoint field method [42–45]. The general assumption in time-difference imaging is that the conductivity variations are small about a background value (low contrast). Furthermore, it is not uncommon to ignore the state evolution model in certain low contrast imaging problems. Here, we pre-compute the Jacobian matrix at conductivity value $\sigma[0] = 1$. The observation matrix in the measurement equation (4) is set such that $J_k = J(\sigma_0), \forall k \in \{1, ..., N_t\}$. Where $N_t$ denotes the last time index.

3.1 CRLB for the unbiased estimator

If the conductivity varies slowly as a function of time and the perturbations are small about a background value, then one may be able to exclude the state evolution model. Furthermore, when the parameters are considered to be deterministic, then Cramér Rao Lower Bound (CRLB) expresses a lower bound on the variance of any unbiased estimator [4] and is used as benchmarking tool to compare the performance of different unbiased estimators. Denoting the true value of the parameter in the linearized model by $\delta \sigma_i \in \mathbb{R}^N$, the $N \times N$ covariance of the estimation error $\hat{\delta \sigma} - \delta \sigma_i$ for the unknown parameter $\delta \sigma$ is bounded from below by the inverse of the Fisher information matrix [4, 38]. The CRLB inequality is given by

$$C_{\hat{\delta \sigma} - \delta \sigma_i} \geq C_{\delta \sigma} = \mathcal{I}^{-1}$$

(5)

Where $C_{\delta \sigma}$ and $\mathcal{I}$ denote the covariance matrix of the estimate and the Fisher information matrix respectively. The FIM is defined as the unbiased estimate of the Hessian matrix and described by the following expression

$$\mathcal{I} = -\mathbb{E}[\nabla^2_{\delta \sigma} \ln p(\delta V|\delta \sigma)].$$

(6)

where $p(\delta V|\delta \sigma)$ denotes the likelihood function. Moreover, the regularity condition [4] must be satisfied

$$\mathbb{E} \left[ \nabla_{\delta \sigma} \ln p(\delta V|\delta \sigma) \right] = 0; \forall \delta \sigma$$

(7)
In the case that $\delta \mathbf{V} \sim \mathcal{N}(J \delta \sigma, C(\delta \sigma))$, the FIM is given by

$$[\mathcal{I}]_{i,j} = \left[ \frac{\partial \delta \sigma_j}{\partial \delta \sigma_i} \right] \mathcal{C}^{-1}(\delta \sigma) \left[ \frac{\partial \delta \sigma_j}{\partial \delta \sigma_i} \right] + \frac{1}{2} \text{tr} \left[ \mathcal{C}^{-1}(\delta \sigma) \frac{\partial \mathcal{C}^{-1}(\delta \sigma)}{\partial \delta \sigma_i} \left( \frac{\partial \mathcal{C}^{-1}(\delta \sigma)}{\partial \delta \sigma_j} \right) \right]$$

(8)

If $\mathbf{w}$ is not a function of $\delta \sigma$, and $\mathbf{w} \sim \mathcal{N}(0, C_w)$, then $\delta \mathbf{V} \sim \mathcal{N}(J \delta \sigma, C_w)$, and the FIM simplifies to

$$\mathcal{I} = J^T C_w^{-1} J$$

(9)

It is well known that the MLE achieves the CRLB asymptotically [4]. The MLE is asymptotically distributed according to the Gaussian probability density function (PDF) given below by

$$\hat{\delta \sigma} \sim \mathcal{N}(\delta \sigma_\text{MLE}, \mathcal{I}^{-1})$$

(10)

It should be emphasized that the CRLB is a lower bound for unbiased estimators and does not take into account any prior information on the parameters. In the case of ill-conditioned problems, for example, due to a large parameter space, such prior information may be necessary to achieve acceptable performance. This is certainly the case in EIT, since the singular values of $J$ gradually decay to zero and the ratio between the largest and smallest nonzero singular values is large. In a study performed by [46], a technique from detection theory [47] was explored to study the optimal current patterns for distinguishing between two regions of conductivity. The dimensionality was reduced to one by defining a region of interest. Here, we adopt the same approach to allow for an unbiased estimator to be employed. We begin by defining a set consisting of indices of finite elements in the region of interest ROI=$\{i: 1 \leq i \leq N\}$, and a vector

$$r_i = \begin{cases} \Delta_i & \text{if } i \in \text{ROI} \\ 0 & \text{otherwise} \end{cases}$$

(11)

Where $\Delta_i$ denotes the area of the finite element index $i$. It can be easily shown that the CRLB of the region of interest is given by

$$\text{VAR}(\delta \sigma) = \frac{1}{r^T (J^T C_w^{-1} J) r}$$

(12)

### 3.2 Bayesian Cramér Rao Bound

We consider the Bayesian counterpart to the CRB, and we refer to it as the Bayesian Cramér Rao Bound. These bounds have been well studied and documented in the field of nonlinear filtering and tracking [39]. We only state results that are relevant to this study. In a Bayesian setting, the parameter vector $\delta \sigma \in \mathbb{R}^N$ is modeled as a random vector and our knowledge of it’s distribution is encoded in to the probability density function $p(\delta \sigma)$.

If $(\mathbf{w}, \delta \sigma)$ is jointly Gaussian and independent random vectors, $\mathbf{w} \sim \mathcal{N}(0, C_w)$ and $\delta \sigma \sim \mathcal{N}(0, C_{\delta \sigma})$, then the Bayesian Cramér Rao Bound inequality is given by

$$C_{\delta \sigma | \mathbf{V}} \geq C_{\delta \sigma | \mathbf{V}} = \mathcal{I}_B^{-1}$$

(13)

Where $\mathcal{I}_B$, is often referred to as the Bayesian information matrix, and reads as

$$\mathcal{I}_B = (J^T C_w^{-1} J + C_{\delta \sigma}^{-1})$$

(14)

Here, $C_{\delta \sigma | \mathbf{V}}$, denotes the covariance matrix of the posterior density [4] and can be computed by the alternative formula

$$C_{\delta \sigma | \mathbf{V}} = C_{\delta \sigma} - C_{\delta \sigma} J^T (C_w^{-1} J C_{\delta \sigma} + C_w^{-1} J C_{\delta \sigma} J^T) C_{\delta \sigma}^{-1}$$

(15)

It is also well known, that the CRLB for the linear Gaussian filtering problem is equivalent to the Covariance matrix of the Kalman filter [39].

### 4 Numerical Results and Discussions

The finite element mesh used in the study is shown in Fig. 1. The forward model consists of 576 triangular finite elements, 313 nodes and 16 electrodes. Furthermore, all results are based on the adjacent current protocol. Here, we have used a measured noise model, so $C_w$ denotes the sample covariance matrix. The measurement system is discussed in [48]. The data has been obtained from a public website and is discussed in Refs [42, 49]. In this study, $N = 576$, and the total number of available measurements $M = 208$. This implies that the problem is ill-posed.

Recall, that the number of rows of the observation matrix is directly linked to the number of electrodes in the experimental setup. The observation matrix with elements $J_{m,j} = (\partial v_m / \partial \sigma_j) \Delta_i$ is organized as follows: the rows of the matrix are 16 multiples of 13 rows of measurements. The first 13 rows of measurements correspond to the current excitation on electrode pair (1, 2) and the last 13 rows of measurements correspond to the current excitation on electrode pair (1, 16). Moreover, no voltage measurement is performed on the electrodes used for excitation. Here, we attempt to demonstrate how the CRB can provide insight to the impact of electrode positioning with respect to the estimation accuracy.

#### 4.1 CRLB for the unbiased estimator

In order to employ an unbiased estimator, we reduce the dimensionality of the problem by defining regions of interest. We emulate the electrode positioning problem, by defining three regions namely, the left lung, right lung and the heart. The setup is shown in Figure 1. The details of this implementation were discussed in section 3.1. Here, we also seek insight into how the number of electrodes impacts the estimation accuracy. We attempt to emulate this problem by computing the CRB as a function of the number of rows of the observation matrix $J$ as shown in Fig.
2. Furthermore, we choose the logarithmic plot of the variance, to allow for a better comparison of the CRB for different regions of interest. It is clear that the lowest CRB, will correspond to the region which will have the highest estimation accuracy. It can be observed from Fig 2 that initially, the right lung has the lowest variance. This agrees with the underlying physics and is explained by the clockwise positioning of the electrodes. The measurements associated with the current excitations on electrode pairs in the set \{ (1, 2), (2, 3), (3, 4) \ldots (8, 9) \} occupy the first rows of the Jacobian matrix. Furthermore, no significant reduction in CRB is observed, by including rows of the Jacobian matrix beyond the 100th row. In the case, of the left lung, the opposite is observed. The largest reduction in CRB is achieved by including measurements associated with current excitation subset \{ (7, 8), (8, 9), \ldots (16, 1) \}. The measurements associated with these current excitations occupy rows 100 to 208 of the Jacobian matrix. Due, to the symmetry of the problem, the CRB of the left and right lung converge to the same value. Finally, the CRB of the heart region is not reduced significantly after including the first few rows of the Jacobian matrix. One can conclude that the measurements, associated with the remaining current excitations, do not contribute much to the estimation accuracy. This can also be explained by the adjacent current protocol not being a suitable measurement strategy for interrogating regions situated far from the electrode array.

4.2 Bayesian Cramér Rao Bound

Here, we compute the BCRB for the given forward problem (geometry). The prior is assumed to be \( \delta \sigma \sim \mathcal{N}(0, C_{\delta \sigma}) \), with the covariance matrix modeled as a diagonal matrix with the variances computed according to

\[
[C_{\delta \sigma}]_{i,i} = (J^T C_w^{-1} J)^{0.5} 
\]

(16)

This choice for the covariance matrix was previously studied in [16]. It is worth emphasizing that accuracy of the Bayesian methods in parameter estimation rely on the correct choice of the prior model [4, 18]. The results of the BCRB computations are shown in Fig. 3. Moreover, given the choice of the current protocol (adjacent), the results agree with the underlying physics. The regions close to the electrodes have low variance levels. As you move towards, the center of the experimental setup, the CRB increases, which translates to estimation accuracy decreasing.

5 Conclusion

In this study, we have investigated the potential application, of the well established Cramér Rao Bound concept for designing, and assessing the quality of experimental setups in electrical impedance tomography. All information about the forward problem, as well as multi-frequency data, and statistical properties of measurement noise is encoded in the Fisher information matrix. FIM can be easily computed and its inverse is the CRLB. This makes, CRB a desirable metric and effective tool for the designing experimental systems and hardware.

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Figure 3. This figure shows the Bayesian Cramér Rao Bound \([J^T C_{\sigma}^{-1} J + C_{\sigma}^{-1}]^{-1}\), for the specified geometry (forward problem). The CRB computations are based on the adjacent current protocol. The covariance matrix of the prior is chosen to be a diagonal matrix and is computed according to \(C_{\sigma}[i,i] = [J^T C_{\sigma}^{-1} J]^{-0.5}\). The color bar indicates the relative variance level.

References


