BOUNDARY CONDITIONS IN HUMAN MOVEMENT I: CONSTRaining CONSTRAINTS

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ABSTRACT

Optimal control models (distal models) play an important role in understanding the principles of human motor control and bio-inspired robotic applications. A major goal has been to identify the cost function (performance index) of the motor behaviour, which has led to such influential models as the ‘minimum jerk’, ‘minimum torque’, and ‘minimum variance’ hypotheses (inter alia). Less attention has been paid to the boundary conditions (BCs) needed to obtain theoretical optimal solutions. The choice of BCs strongly affects solutions, particularly when cost functions depend on high-order derivatives, as in human movement. To avoid the problem of arbitrary hypothetical constraints it is essential to justify BCs biologically. We examine analytically the effects of different BCs on minimum square derivative trajectories (minimum acceleration; minimum jerk) as simple tractable illustrations. We also examine the difference between physical and neuro-musculo-skeletal constraints, and conclude that it is not possible to justify kinematic models without appealing to non-kinematic BCs. Ultimately, BCs are an empirical issue that need to be measured independent of any theoretical model.

KEYWORDS: Arm movements, eye movements, minimum jerk, Fourier analysis, measurement noise, boundary conditions

INTRODUCTION

Distal models of human behaviour attempt to model why movements occur the way they do by focussing on the control objectives of movement and the evolutionary selective pressures that maximise fitness (distal causes). In contrast, proximal models attempt to explain how movements are generated (the neuro-bio-physics). A key approach to the distal problem has been the assumption that natural selection ‘improves’ behavioural strategies until some optimum is reached. Consequently, finding Nature’s cost function has been the major goal of the distal approach. An important issue in such models is the problem of ‘arbitrary hypothetical constraint’ [1], where constraints are employed to find theoretical optimal solutions, but have little or no biological justification. To escape circularity, a core requirement of any distal model is to base all assumptions in empirical observation, or to show they are optimal themselves.

Distal models have been particular influential in the study of human reaching and are finding increasing application in ‘biologically inspired’ engineering. Optimising strategies such as ‘minimum jerk’ [2], ‘minimum torque’ [3], and ‘minimum variance’ [4] have received much attention, but numerous other cost functions have been proposed. Although there may be disagreement over the precise form of the cost function, there is little doubt that it is sensitive to high order derivatives of effector position, such as jerk (3rd derivative of position) or snap (4th derivative). It has been endlessly argued that this leads to smooth trajectories [5] (the MJ cost function was even conceived for this purpose). However, this theoretical smoothness is a direct result of the boundary conditions (i.e. the type of discontinuity permitted). Empirical trajectories could in principle be much smoother than those observed. Indeed we have argued that trajectories are probably the least smooth possible! Moreover, some movements have very rapid onsets with no resemblance to the classic MJ profile, such as articulatory speech movements [6]. Are these behaviours also optimal?

It is clear that boundary conditions (BCs) and discontinuities are a key issue in distal models. In this paper we illustrate and attempt to clarify the effect of different BCs. What is their biology, and how do we avoid the circularity of arbitrary hypothetical constraints? We focus on minimum square derivative (MSD) optimal trajectories (minimum acceleration, jerk, snap etc.), which are optimal trajectories for cost functions that are the integral of the square of a derivative over the movement...
duration. Although MSD profiles do not perfectly match observed data, they do provide a reasonable fit, and have received much attention in robotics. MSD profiles are also limiting optimal solutions for other cost functions (such as minimum variance) for very brief movements.

AN EMPIRICAL EXAMPLE

We first consider an illustrative example of fast point-to-point movements recorded from human subjects. We measured the time course of the movement of the fingertip in human subjects with no physical constraints on the arm, shoulder or trunk. Subjects were asked to make rapid point-to-point movements horizontally in the plane of a horizontal work surface in 2 conditions: 1) with the finger starting and ending about 2 cm above the surface, so that the finger never made contact with the surface; 2) as before but with the fingertip starting and ending with touching the surface (but not during the movement). We used a tether-less infrared photodiode system (CODA MPX30, Charnwood dynamic: http://www.charndyn.com, sampling rate 400Hz; resolution ~0.1 mm.)

In the non-touching condition we observe typical bell-shaped velocity trajectories (Fig. 1a). When the movement starts and ends with the finger touching the surface (but not touching during the movement) we find more responses with rapid changes in velocity at the onset and the offset of the movement (Fig. 1b). Note that plots are from filtered data, so it is not clear how abrupt the changes are [6].

This very simple task demonstrates that changing the physics of the situation (touching or not touching) leads to different trajectories. From a proximal viewpoint, we might argue that the abrupt jump in velocity is due to the interaction among static friction, dynamic friction, and the downward (normal) force. Thus, initially the subject presses down increasing both frictions, but as the downward pressure is reduced friction is overcome and the movement is launched, leading to a rapid increase in lateral velocity. Presumably, the process is reversed upon landing. Clearly, we could pursue this problem in depth experimentally to find a biophysical (and proximal) explanation

The distal approach is quite different. In the non-touching condition we find the typical ‘smooth’ bell-shaped profiles, which have been considered optimal. So why do we observe a different strategy in the touching condition? Which trajectory is cheaper? It seems unlikely that the abrupt changes would be included in the cost function because high order derivatives will be huge (if not infinite in the limit). If discontinuities were free, should they be more frequent if not ubiquitous?

OPTIMALITY (DISTAL) MODELS

To address the issue of BCs and discontinuities, we first define costs on the open interval \(0 < t < T\) using standard variational calculus. (All problems discussed here can be recast as a set of 1st order differential equations and solved using Pontryagin’s maximum principle to yield the same results.) We consider the 1-D optimization problem where the ‘open’ cost \(J_0\) is given by the functional:

\[
J_0 = \lim_{\varepsilon \to 0} \int_0^{T-\varepsilon} L(x(t), x'(t)) dt
\]

where the Lagrangian \(L(\cdot)\), describes how cost depends on the position, \(x(t)\), and its derivatives \(x^{(n)}(t) = d^n x(t)/dt^n\), of the biological effector (e.g. finger, eye etc.). The optimal movement trajectory \(x^*(t)\) is the trajectory that minimises \(J_0\), and can be found by solving the Euler-Lagrange equation:

\[
\frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial x'} \right) + \cdots + (-1)^n \frac{d^n}{dt^n} \left( \frac{\partial L}{\partial x^{(n)}} \right) = 0
\]

which yields a 2n-order differential equation, with 2n unknown parameters. The traditional approach has been to set enough lower order derivatives to zero at the beginning and end of the movement:

\[
x^{(j)}(0) = 0, \quad x^{(j)}(T) = 0, \quad 1 \leq j \leq 2n - 2
\]

and the initial and final positions to be:

\[
x(0) = \mathbf{x_0}, \quad x(T) = \mathbf{x}_T
\]

Together (3) and (4) provide 2n equations needed to constrain (2). However, apart from mathematical convenience, it is not clear why there should be just 2n BCs. It is quite feasible to find optimal solutions in the under constrained problem: we simply find the values of unconstrained parameters that minimise total cost. Even over-constrained problems may have a minimum,
MINIMUM SQUARE DERIVATIVES

The $n$th order Minimum Square Derivative functions (MSD) is the optimal function when the Lagrangian is simply the square of a single derivative

$$L(t) = \left(x^{(n)}(t)\right)^2.$$  \hfill (7)

From (2), the solution is given by a $2n-1$ order polynomial where the coefficients are determined from the boundary conditions, if any.

Minimum Acceleration Example

We first consider the simplest example, where

$$L(t) = \left(x^{(2)}(t)\right)^2.$$  \hfill (8)

The general solution is given by a $3^{rd}$ order polynomial. Applying (4), we have:

$$x^{*}(t) = a_3t^3 + a_2t^2 + a_1t.$$  \hfill (9)

with

$$x(0) = 0, \quad a_3T^3 + a_2T^2 + a_1T = A.$$  \hfill (10)

Assuming all derivatives are zero outside the closed interval, velocity and acceleration on the open interval are given by:

$$x^{*}(1) = 3a_3t^2 + 2a_2t + a_1.$$  \hfill (11)

$$x^{*}(2) = 6a_3t + 2a_2.$$  \hfill (12)

Substituting into (1), we obtain the total open cost:

$$J_o = 36a_3^2T^5 / 5 + 8a_3a_2a_1T^4 + 4a_2^2T.$$  \hfill (13)

On the closed interval, however, we note that velocity is generally discontinuous at $t=0$ and $t=T$. Denoting the discontinuity costs as quadratic by $C_k(h,t) = h^2$ and weighting them by arbitrary positive constants, $w_1$ and

$$J = J_o + w_1a_1^2 + w_2(a_1 + 2a_2T + 3a_3T^2)^2.$$  \hfill (14)

To find the minimum subject to the constraint that endpoint position is $A$, we solve the set of 3 simultaneous equations adjoined by (10):

$$\frac{\partial}{\partial a_j} \left[J + \lambda(a_3T^3 + a_2T^2 + a_1T)\right] = 0 \quad j=1,2,3.$$  \hfill (15)

to specify admissible functions for a well-posed problem.

where $\lambda$ is a constant Lagrange multiplier. This yields:

$$a_1 = 6\lambda(2 + w_2T) / D$$  \hfill (16a)

$$a_2 = 3w_1A(2 + w_2T) / D$$  \hfill (16b)

$$a_3 = -2A(w_1 + w_2T + w_3) / TD$$  \hfill (16c)

where $D = T(12 + 4T(w_1 + w_2) + w_1w_2T^2)$.

If there are no penalties for discontinuities, then the optimal solution is constant velocity and the total cost is zero (Fig. 2). As the $w$ increase, the optimal discontinuity strengths decrease and the trajectory becomes more curvilinear. Asymmetrical costs lead to asymmetrical velocity trajectories. If discontinuities are forbidden (i.e. infinite cost), then the optimal trajectory becomes parabolic. This is the profile usually presented as ‘minimum acceleration‘, but of course, all trajectories that satisfy (9) and (16) are minimum acceleration profiles.

Minimum Jerk example

For the minimum jerk problem, the Lagrangian is $L(t) = (x^{(3)}(t))^2$ and the optimal trajectory is a $5^{th}$ order polynomial: $x^{*}(t) = a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t$. Solving for minimum jerk is similar, although we need to consider costs for higher order discontinuities. Explicit analytic solutions can be found, but they will not be printed here as they are extremely cumbersome [A Matlab Maple routine can be supplied upon request]. Plots of optimal trajectories are given in figure 2.

Figure 2: Examples of different discontinuity costs on minimum acceleration (left column) and minimum jerk (right column) trajectories (normalised to $T = 1$). 1st row: optimal trajectories with free discontinuities (no cost) showing optimal trajectory with constant velocity; 2nd row: high cost on end-point velocity discontinuity; 3rd row: moderate costs on initial and final velocity discontinuities; 4th row: infinite costs on initial and final velocity discontinuities yielding classic trajectory shapes.
ORIGINS OF BCS

Thus far, we have seen that attaching costs to discontinuities can drastically alter the shape of the optimal trajectory for a given Lagrangian. We now return to our empirical reaching example (Fig. 1) where for the sake of argument we shall assume that trajectories are optimal with a MJ Lagrangian (to be discussed below). We would need to infer that discontinuity costs are sufficiently high in the non-touching condition to preclude any discontinuities. In the touching condition discontinuity costs would have to be much lower, but not zero as we do not observe constant velocity profiles. To explain this, we consider two sources of BC constraints: 1) those due to low pass characteristics of musculo-skeletal dynamics (plant); 2) those due to physics of the system.

Neuromuscular BCs

Assume that the motor plant can be described as a linear system of order m, and denote ut as the aggregate motor command, then position is constrained by the differential equation:

\[ c_m x^{(m)} + c_{m-1} x^{(m-1)} + \ldots + c_1 x = u(t) . \]  

(17)

If we assume that the motor command can rise no faster than a step function, all derivatives up to m-1 must be continuous. Only the mth derivative can be discontinuous if the control is discontinuous. Since effectors have mass/inertia, the lowest possible discontinuity could be in acceleration (although it may be higher). Therefore, trajectories could only be MJ if the system were 3rd order; m=3 in (17). It seems a remarkable coincidence that a 3rd order plant is needed to explain a MSD of 3rd order! A more parsimonious interpretation of MJ profiles approach is to consider a new Lagrangian of the form:

\[ L(t) = s(t)(u(t))^2 \]  

(18)

where ut is the motor command governed by (17), and st is some smooth positive-definite weighting function included for generality. For very brief movements, the open cost function becomes:

\[ \lim_{T \to 0} \frac{1}{T} \int_0^T s(t) u(t)^2 dt \]

(19)

provided s(0+) is sufficiently smooth. Thus MSD trajectories can be viewed as limiting optimal trajectories for cost functions of the form given by (18). For example, the minimum variance cost function belongs to this family [4]. It should be noted that the MJ model is now dynamic rather than kinematic.

Physical BCs

The above argument implies that velocity and acceleration discontinuities should be impossible. Yet, we observe very sudden changes in velocity in some movements (fig.1b in ref. 6). It seems likely that these discontinuities reflect the physics of the situation, rather than a remarkable neural exception to (17). It is an empirical observation that for rough surfaces, the coefficient of static friction exceeds kinetic friction. Thus, when touching the surface, the horizontal force must overcome static friction, but once movement commences friction suddenly decreases leading to a step (or very rapid) increase in force that could not be achieved by neural dynamics alone. The size of the step (strength of discontinuity) will depend on the applied downward (normal) force. Thus, by suitable timing of lateral and vertical forces, it would be possible to generate a step change in horizontal force as well as lifting off the surface. The procedure could be reversed on landing. There are other physical mechanisms that can occur, such as colliding with an object or another effector (e.g. bringing lips together) which would result in velocity discontinuities. The reverse may occur in explosive separation of effectors as in plosive speech movements [6].

These processes allow lower order discontinuities to occur, and reduce cost in the open interval (assuming the Lagrangian depends on high order derivatives). However, we expect the advantages to be balanced by costs that might be incurred by such strategies, such as mechanical work done, errors in timing, damage to effector, etc. Exploration of these processes are beyond the scope of this paper.

DISCUSSION

Boundary conditions (BCs) are a fundamental part of any optimality (distal) model of behaviour. To avoid circularity (the ‘arbitrary hypothetical constraint’ [1]), BCs need to be justified rather than just assumed. For example, the claim that rapid reaching movements are fitted by the traditional ‘minimum jerk’ [2] is quite arbitrary unless it can be justified why initial and final velocities and accelerations should be zero. The same argument applies to any other distal model. However, this raises many questions.

An important one is the problem of measuring BCs from empirical data. We have considered discontinuities in the limiting mathematical sense, but this is not possible to measure because of finite bandwidth and noise. Fourier analysis seems a promising numerical/analytical technique [6], and an accompanying paper shows that discontinuities can only be established empirically up to some maximum Fourier frequency [7].

Given that we can measure discontinuities with some precision using the Fourier approach, a major question is whether point-to-point reaching movements can really be fit by a specific distal model, such as the traditional MJ profile. In a second accompanying paper we show that non-touching trajectories are not MJ with traditional BCs (zero initial and final velocities and accelerations). A dynamic model (such as minimum variance) provides a better description.

Of foremost interest is whether trajectories with lower order discontinuities are also optimal. That is, are the
discontinuities evaluated in the optimal control of a movement? Is the physics exploited deliberately or just a perturbation to a pre-existing control strategy? This may be a fundamental test of any distal model.

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REFERENCES