APPLICATION OF A LINEAR TIME-VARYING CONTROL TECHNIQUE TO TRAJECTORY STABILIZATION FOR NONLINEAR MULTI-INPUT SYSTEMS

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ABSTRACT
In this paper, we consider the stabilization controller for some desired trajectory of nonlinear systems. For such a problem, it should be one of the best strategies to apply the linear time-varying controller for linear approximate model around the trajectory. However, because a controller design procedure for linear time-varying system is not necessarily simple, some other methods like various types of nonlinear controllers or a gain scheduling control based on the linear time-invariant controller are commonly used.

The author proposed the simple design procedure of pole placement controller for linear time-varying multi-input systems. The feedback gain can be obtained directly from the plant parameters without transforming the system into any standard form. This paper show the applicability of such a controller to linear time-varying approximate model around some desired trajectory of nonlinear systems.

KEY WORDS
stabilization of trajectory: nonlinear system: linear time-varying system: pole placement

1 Introduction
This paper concerns a stabilization problem of some desired trajectory of nonlinear systems. The most natural and effective method might be to approximate the nonlinear system around this trajectory by a linear time-varying system and then to apply a linear time-varying stabilizing controller to this system. This method can be applied wide class of nonlinear systems, and, because we make use of the approximate model, controllers have some robustness properties in it to some extent. However, since, design procedure of linear time-varying system is not necessarily simple enough [1] [2] [3], some other methods are often used for this problem. Various types of nonlinear controllers[4][5] are such choices. But, to apply the nonlinear control technique, the type of nonlinearity is restricted. And, because the most of such methods are based on the cancellation of nonlinearity of the system, we should know the precise presentation of nonlinearity of the system to be controlled. Other option may be the gain scheduling techniques. These methods are generally based on multiple time-invariant controllers, and we control the system by switching them. This method may be a simple substituted method until we will have highly applicable nonlinear or time-varying controller design techniques.

From the above point of view, the author et. al. have proposed simple pole placement controller design method [6][7]. Such controller is obtained by finding a new output signal so that the relative degree from the input to this new output is equal to the system degree. We do not need to transform the system into any standard form for the controller design. In this paper, we show the applicability of such a pole placement controller design procedure to the problem of the stabilization of some desired trajectory of nonlinear systems. Section 2 will present how to design the pole placement controller for linear time-varying multi-input systems. Then, Section 3 will show the applicability of linear time-varying pole placement controller to the stabilization of some desired trajectory of a nonlinear multi-input system. Using a numerical example, its applicability and robustness will be shown.

2 Pole Placement of Multi-Input Systems
In this section, the design procedure of the pole placement control for linear multi-input time-varying system is summarized.

Consider the following linear time-varying system.

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) \] (1)

where, \( x(t) \in R^n \) and \( u(t) \in R^m \) are the state variable and the input signal vectors. \( A(t) \in R^{n \times n} \) and \( B(t) \in R^{n \times m} \) are the time varying coefficient matrices, which are smooth functions of \( t \). Using vectors \( b_i(t) \in R^n \), the matrix \( B(t) \) can be written as follows.

\[ B(t) = \begin{bmatrix} b_1(t) & b_2(t) & \cdots & b_m(t) \end{bmatrix} \] (2)

Let \( B_j(t) \in R^{n \times m} \) be defined as follows.

\[ B_0(t) = B(t) \]
\[ B_{j+1} = A(t)B_j(t) - \dot{B}_j(t) \] (3)
Then, the controllability matrix, $U_C(t)$, is defined by

$$U_C(t) = \begin{bmatrix} B_0(t) & B_1(t) & \cdots & B_{n-1}(t) \end{bmatrix}$$

where $b_k^i(t)$ is the $k$-th column vector of $B_i(t)$. Hence, from (3), $b_k^i(t)$ satisfies the same equations, i.e.,

$$b_k^0(t) = b_k(t)$$
$$b_k^{i+1}(t) = A(t)b_k^i(t) - \dot{b}_k^i(t)$$
$$k = 1, 2, \cdots, m \quad i = 0, 1, 2, \cdots$$

The system (1) is controllable if and only if

$$\text{rank}U_C(t) = n. \quad (6)$$

For controllable systems, the controllability indices $\mu_i$ ($i = 1, \cdots, m$) are defined which satisfy the following equations.

$$\text{rank}R(t) = n$$
$$\sum_{i=1}^{m} \mu_i = n \quad (7)$$

where

$$R(t) = \begin{bmatrix} b_0^0(t) & b_0^1(t) & \cdots & b_0^{\mu_0-1}(t) \\ \vdots & \vdots & \ddots & \vdots \\ b_m^0(t) & b_m^1(t) & \cdots & b_m^{\mu_m-1}(t) \end{bmatrix}. \quad (8)$$

In this section, the system (1) is assumed to be controllable that has constant controllability indices. Such a system is called a lexicographically-fixed system. It is also assumed that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ without loss of generality.

The problem is to find the state feedback

$$u(t) = K(t)x(t) \quad (9)$$

which makes the closed loop system equivalent to some time invariant linear system with arbitrarily stable poles.

The simplified desiging procedure of the pole placement control for linear time-varying multi-input system is as follows. [6]

(The Design Procedure of Pole Placement)

[STEP 1] Using the equations (4) and (5), i.e.,

$$b_k^0(t) = b_k(t)$$
$$b_k^{i+1}(t) = A(t)b_k^i(t) - \dot{b}_k^i(t)$$
$$k = 1, 2, \cdots, m \quad i = 0, 1, 2, \cdots$$

and

$$U_C(t) = \begin{bmatrix} b_0^1(t) & b_0^m(t) \\ \vdots & \vdots \\ b_1^{m-1}(t) & b_m^{m-1}(t) \end{bmatrix} \quad (11)$$

check the controllability of the system (1) and find the controllability indices $\mu_i$. Then, define the matrix, $R(t)$ by (8), i.e.,

$$R(t) = \begin{bmatrix} b_0^0(t) & b_0^1(t) & \cdots & b_0^{\mu_0-1}(t) \\ \vdots & \vdots & \ddots & \vdots \\ b_m^0(t) & b_m^1(t) & \cdots & b_m^{\mu_m-1}(t) \end{bmatrix}. \quad (12)$$

[STEP 2] Calculate $C(t) \in R^{m \times n}$ by

$$C(t) = \begin{bmatrix} c_1(t) \\ \vdots \\ c_m(t) \end{bmatrix} = WR^{-1}(t). \quad (13)$$

where

$$W = \text{diag}(w_1, w_2, \cdots, w_m) \quad (14)$$

and

$$w_i = [0, \cdots, 0, 1] \in R^{1 \times \mu_i} \quad (i = 1, \cdots, m) \quad (15)$$

Here, $c_i(t) \in R^{1 \times m}$. And, then, calculate $c_i^j(t) \in R^{1 \times m}$ by the following equations.

$$c_i^0(t) = c_i(t)$$
$$c_i^1(t) = c_i^0(t)A(t) + c_i^0(t)$$
$$\vdots$$

$$c_i^{\mu_i}(t) = c_i^{\mu_i-1}(t)A(t) + c_i^{\mu_i-1}(t)$$
$$(i = 1, \cdots, m) \quad (16)$$

[STEP 3] Determine $m$ desired stable characteristic polynomials by

$$\alpha^i(s) = s^{\mu_i} + \alpha^{\mu_i-1}s^{\mu_i-1} + \cdots + \alpha^0 \quad (17)$$

where $s$ is a differential operator.

From STEP 1 to STEP 3, it should be noted that if we define a new output signal $y \in R^n$ by

$$y = C(t)x$$

the total relative degree from $u$ to this new output, $y$, becomes the system degree, $n$. And, that we have the following equation.

$$\begin{bmatrix} \alpha^1(s) \\ \vdots \\ \alpha^m(s) \end{bmatrix} y(t) = D(t)x(t) + \Lambda(t)u(t) \quad (18)$$

Here,

$$D(t) = \begin{bmatrix} D_1(t) \\ D_2(t) \\ \vdots \\ D_m(t) \end{bmatrix}, \quad \Lambda(t) = \begin{bmatrix} \Lambda_1(t) \\ \Lambda_2(t) \\ \vdots \\ \Lambda_m(t) \end{bmatrix} \quad (19)$$
Following the equation as a closed loop.

\[ \dot{x}(t) = (A(t) - B(t)\Lambda(t))x(t) \] (24)

Let \( T(t) \) be the time varying matrix defined by

\[
T(t) = \begin{bmatrix}
  c^0_i(t) \\
  \vdots \\
  c^{\mu_i-1}_i(t) \\
  \vdots \\
  c^0_m(t) \\
  \vdots \\
  c^{\mu_m-1}_m(t)
\end{bmatrix}
\] (25)

and define the new state variable \( w \) by

\[
w(t) = T(t)x(t), \quad w = \begin{bmatrix}
y_1(t) \\
  \vdots \\
  y_{\mu_i-1}(t) \\
  \vdots \\
y_m(t) \\
  \vdots \\
  y_{\mu_m-1}(t)
\end{bmatrix}
\] (26)

Then, (24) is transformed into

\[
\dot{w} = \{T(t)(A(t) - B(t)D(t))T^{-1}(t) - T(t)\hat{T}^{-1}(t)\}w
\]

\[
= \begin{bmatrix}
  A^*_1 & 0 \\
  \vdots & \ddots \\
  0 & A^*_m
\end{bmatrix}w
\] (27)

where

\[
A^*_i = \begin{bmatrix}
  0 & 1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  \vdots & & \ddots & \vdots \\
  -\alpha^0_i & \cdots & \cdots & -\alpha^i_{\mu_i-1}
\end{bmatrix} 
\] (28)

which is the realization of (23). This implies that the closed loop system is equivalent to the time invariant linear system which has the desired closed loop poles. \( (\det(sI - A^*) = \alpha^1(s) \cdot \alpha^2(s) \cdots \alpha^m(s)) \)

The non-singularity of \( T(t) \) is guaranteed by the following Theorem.

**Theorem 1** If the system (1) is controllable, then, the matrix for the change of variable, \( T(t) \), given by (25) is nonsingular for all \( t \).

This theorem can be proved by straightforward calculation as for the time invariant case.

It is well known that the exponential stability is preserved between two equivalent linear time-varying systems if the transformation matrix is Lyapunov transformation [8] [9]. For the matrix \( T(t) \) to be a Lyapunov transformation matrix, it should be nonsingular and both of \( T(t) \) and \( T^{-1}(t) \) should be continuous and bounded for all \( t \).

Then, to guarantee the stability of the closed loop system, we need the following Theorem.

**Theorem 2** In the above pole placement control, the closed loop system is exponentially stable if the transformation matrix \( T(t) \) in (25) is Lyapunov transformation.

### 3 Stabilization of a Trajectory for Nonlinear Systems

In this section, we consider the stabilization problem of some particular trajectory of a nonlinear system. For this purpose, we approximate the nonlinear system by using a linear time-varying system along some desired trajectory. And then, the simplified pole placement controller design procedure will be applied to stabilize this trajectory.

Consider the following nonlinear system.

\[
\dot{x}(t) = f(x(t), u(t))
\] (29)

Here, \( x(t) \in R^n \) and \( u(t) \in R^l \) are the state vector and the input signal. Let \( x^*(t) \) and \( u^*(t) \) be some desired trajectory and desired input signal, that is,

\[
\dot{x}^*(t) = f(x^*(t), u^*(t)), \quad x^*(0) = x^*_0
\] (30)

where the initial state, \( x^*_0 \) is on the desired trajectory. Let \( \Delta x(t) \) and \( \Delta u(t) \) be defined by

\[
\Delta x(t) = x(t) - x^*(t)
\]

\[
\Delta u(t) = u(t) - u^*(t)
\] (31)
Then, (30) can be approximated by the following linear time-varying system around along \((x^*(t), u^*(t))\).

\[
\Delta \dot{x}(t) = A(t) x(t) + B(t) \Delta u(t)
\]

(32)

where

\[
A(t) = \frac{\partial}{\partial x} f(x^*(t), u^*(t))
\]

\[
B(t) = \frac{\partial}{\partial u} f(x^*(t), u^*(t))
\]

(33)

Here, \(A(t)\) and \(B(t)\) may be known as their explicit form of functions or known as numerical data. In any case, the pole placement controller design procedure can be applied to stabilize \(\Delta x(t)\).

**[EXAMPLE]** Consider the following non-linear system with two inputs.

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= 0.5(1-x_1^2)x_2 - x_1x_3 + u_1 \\
\dot{x}_3 &= x_1^2 - x_3 + u_1 + u_2
\end{align*}
\]

(34)

Let the desired trajectory for this system be

\[
x_1^*(t) = \cos t, \quad x_2^*(t) = -\sin t, \quad x_3^*(t) = 1
\]

(35)

Then, the desired open loop input \(u^*\) becomes

\[
\begin{align*}
u_1^*(t) &= -2 \cos t - 0.5(1 - \cos^2 t) \sin t \\
u_2^*(t) &= -\cos^2 t + 1 - u_1^*(t)
\end{align*}
\]

(36)

Fig.1 and Fig.2 show the desired input and the desired state response. The initial state is on this desired trajectory, that is, \(x_1(0) = 1, x_2(0) = 0\) and \(x_3(0) = 1\). When we add a disturbance signal shown in Fig.3 to the desired input \(u_1^*(t)\), the disturbed state response becomes as shown in Fig.4 for the same initial condition.
where

\[
A(t) = \begin{bmatrix}
0 & 1 & 0 \\
-x_1^2(t) - x_2^2(t) & 0.5(1 - x_1^2(t)) & 0 \\
2x_1(t) & 0 & -x_1(t)
\end{bmatrix}
\]

and,

\[
B(t) = \begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}
\]

From (13),

\[
\Delta x_i(t) = x_i(t) - x_i^*(t) \quad (i = 1, 2, 3)
\]

\[
\Delta u_i(t) = u_i(t) - u_i^*(t) \quad (i = 1, 2)
\]

The pole placement state feedback for this system is obtained through the following steps:

[STEP 1] By calculating the controllability matrix \( U_c \) using (10) and (11), we have

\[
\mu_1 = 2, \quad \mu_2 = 1
\]

and

\[
R(t) = \begin{bmatrix}
0 & 1 & 0 \\
1 & d(t) & 0 \\
1 & -1 & 1
\end{bmatrix}
\]

where

\[
d(t) = 0.5(1 - \cos^2 t) - \cos t
\]

[STEP 2] From (13), \( C(t) \in R^{2 \times 3} \) is obtained as follows.

\[
C(t) = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
1 & d(t) & 0 \\
1 & -1 & 1
\end{bmatrix}^{-1}
\]

Then, according to (16), we have

\[
c_1^0(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]

\[
c_1^1(t) = \begin{bmatrix} 0 & 1 \end{bmatrix}
\]

\[
c_2^0(t) = \begin{bmatrix} \cos t \sin t - 1, & 0.5(1 - \cos^2 t), & -\cos t \end{bmatrix}
\]

\[
c_2^1(t) = \begin{bmatrix} d(t) + 1 & -1 & 1 \end{bmatrix}
\]

\[
c_2^2(t) = \begin{bmatrix} d(t) + 2 \cos t - \cos t \sin t + 1, & 1 + \cos t, & \cos t - 1 \end{bmatrix}
\]

[STEP 3] Let the desired stable characteristic polynomials, \( \alpha_1(s) \) and \( \alpha_2(s) \), be chosen as

\[
\alpha_1(s) = s^2 + 4s + 3
\]

\[
\alpha_2(s) = s + 2
\]

from which, using (20), we have

\[
D(t) = \begin{bmatrix}
2 + \cos t \sin t, & d(t) + 2d(t) + 2\cos t - \cos t \sin t + 3, & 4.5 - \cos^2 t, \cos t \\
& & \cos t - 1, \cos t + 1
\end{bmatrix}
\]

\[
\Lambda(t) = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

[STEP 4] From the above,

\[
\Delta u(t) = -\Lambda^{-1}(t)D(t)\Delta x(t)
\]

which implies the trajectory stabilizing control input is

\[
u(t) = u^*(t) - \Lambda^{-1}(t)D(t)\Delta x(t)
\]

More precisely,

\[
u_1(t) = u_1^*(t) - (2 + \cos t \sin t)\delta x_1(t)
\]

\[
-(4.5 - \cos^2 t)\Delta x_2(t) - \cos t \Delta x_3(t)
\]

\[
u_2(t) = u_2^*(t)
\]

\[-(3 + 2d(t) + d(t) + 2\cos t - \cos t \sin t)\Delta x_1(t)
\]

\[+(1 - \cos t)\Delta x_2(t) - (3 + \cos t)\Delta x_3(t)
\]

The feedback control \( \Delta u(t) \) is shown in Fig.5, and, the closed loop state response is shown in Fig.6 with the same disturbance and the same initial condition as previous two simulations.

![Figure 5. Feedback Control Input Δu₁ and Δu₂](image)

![Figure 6. Closed Loop State Response](image)

The controller works very well around the trajectory against disturbance. Next, to show the robustness of this controller, we replace the plant equation (34) by the following perturbed system.
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= 0.5(1-x_1^2)x_2 - x_1 x_3 + 0.3(x_1 + x_2 x_3) + u_1 \\
\dot{x}_3 &= x_1^2 - 0.8x_3 + 0.3x_2 + u_1 + u_2 \quad (41)
\end{align*}

\text{Figure 7. State Response of the Perturbed System (41) to the Desired Input (36) with the Initial Condition } x_1(0) = x_3(0) = x_3(0) = 0.5

\text{Figure 8. State Response of the Perturbed System with Feedback Control}

When we put the desired input (35) into the perturbed system (41) under the condition \( x_1(0) = x_3(0) = x_3(0) = 0.5 \), the state response becomes unstable as shown in Fig.7.

Fig.8 shows the state response of the perturbed system (41) with the same state feedback for the original system (34) \( x_1(0) = x_3(0) = x_3(0) = 0.5 \). The state error between the desired trajectory (35) and the perturbed system response with the feedback is shown in Fig.9, and the feedback input is shown in Fig.10. This shows that this type of trajectory stabilizing controller has a robustness property for the perturbation of the nonlinear system.

It should be noted that, in the controller design procedure, we use the symbolic calculation here. However, it is also possible to use only the numerical calculation for the design procedure, which makes this design method more valuable.

\section{4 Conclusion}

This paper concerned the problem of stabilization of some particular trajectory of nonlinear systems. Nonlinear system can be approximated using a linear time-varying system along a certain trajectory. The author already proposed the simple design procedure of the pole placement controller for linear time-varying system. The paper showed that this design method can be applied to the trajectory stabilization control of nonlinear systems.

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig7}
\caption{State Error}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig10}
\caption{Feedback Control Input \( \Delta u_1 \) and \( \Delta u_2 \) for the Perturbed System}
\end{figure}

\textbf{References}


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